Solutions to Atiyah and MacDonald's *Introduction to Commutative Algebra*

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Chapter 1

Rings and Ideals

1.1

We see that $x \in \mathfrak{R}$ implies $x \in \mathfrak{J}$ (the Jacobson radical), hence $1 + xA \subset A^\times$. In particular, $1 + x$ is a unit. We can now easily deduce that the sum of a nilpotent element and a unit is a unit itself.

1.2

We have the following:

(i) If $f(x) = a_0 + a_1x + \ldots + a_nx^n$ is a unit in $A[x]$, let $g(x) = b_0 + b_1x + \ldots + b_mx^m$ be its inverse. We deduce that a_0 is a unit. We use induction on n to prove that the coefficients are nilpotent. The case $n = 0$ is a tautology. If the proposition holds for $n - 1$, then we see that $a_n^{r+1}b_{m-r} = 0$ (we just write down explicitly the relations that ensue from $fg = 1$ and then multiply each of them by increasing powers of a_n). In particular, this implies that $a_n^{m+1}b_0 = 0$ and, since b_0 is a unit, we deduce that $a_n^{m+1} = 0$. Hence a_n is nilpotent and we may apply the inductive hypothesis.

The converse follows from exercise 1 and exercise 2, (ii).

(ii) If $f(x)$ is nilpotent, then we can apply induction to n to show that all its coefficients are nilpotent. The case $n = 0$ is a tautology. In the general case, it's apparent that the leading coefficient will be a_n^m for suitable $m \in \mathbb{N}$ hence a_n is nilpotent. Now the inductive hypothesis applies.

Conversely, if all the coefficients of $f(x)$ are nilpotent and $d \in \mathbb{N}$ is such that $a_i^d = 0$, $0 \leq i \leq n$ (e.g. let d be the sum of the orders of all the orders of the coefficients), then we see that $f(x)^d = 0$.

(iii) If f is a zero divisor, then let g be a polynomial of *minimal* order, such that $fg = 0$. If $g = b_0 + b_1x + \ldots + b_mx^m$ is not of degree 0, then $a_nb_m = 0$, hence a_ng is annihilates f but is of degree less than m, contradiction. Therefore, g is of degree 0; there is $a \in A$, such that $af = 0$. The converse is trivial.

(iv) If fg is primitive, then so are f and g, too. The converse is just Gauss's Lemma, in a more general context (the elementary argument still carries though).

1.3

The generalization follows easily by induction.

1.4

If $\mathfrak{J}$ denotes the Jacobson radical and $\mathfrak{R}$ denotes the nilpotent radical, then $\mathfrak{J} \supset \mathfrak{R}$, since $\mathfrak{R}$ is the intersection of all prime ideals, while $\mathfrak{J}$ is the intersection of all prime and maximal ideals. Therefore, we only need to show $\mathfrak{J} \subset \mathfrak{R}$ in $A[x]$. Indeed, if $f(x) \in \mathfrak{J}$, then $1 - f(x)g(x) \in A^\times$, for all $g(x) \in A[x]$. In particular $1 - f(x)x$ is a unit, hence if $f(x) = a_0 + a_1x + \ldots + a_nx^n$, then $a_0, a_1, \ldots a_n$ are all nilpotent, hence by exercise 2, (ii) $f(x) \in \mathfrak{R}$. This completes the proof.

1.5

We have the following:

(i) If $f = \sum_{n=0}^{\infty} a_n x^n \in A[[x]]$ is invertible, then obviously a_0 is a unit. Conversely, if a_0 is a unit, then we may let b_0 be such that $a_0 b_0 = 1$ and then we may define b_n, $n \in \mathbb{N}$ recursively by the explicit relations they have to satisfy.

(ii) If $f \in A[[x]]$ is nilpotent, then so is $a_0 \in A$, which, raised to a power, is the constant term when f is raised to a power. Therefore, $f - a_0 = xg(x)$, $g(x) \in A[[x]]$, will also be nilpotent, hence $g(x)$ will also be nilpotent. But the constant term of $g(x)$ is a_1, which must be nilpotent, too. By this process, we show that all the coefficients will have to be nilpotent. The inverse is not true; a sufficient condition for it to be true would be the ring to be Noetherian.

(iii) We easily see that $1 - f(x)g(x)$ is a unit for all $g(x) \in A[[x]]$ if and only if $1 - a_0 b_0$ is a unit for all $b_0 \in A$, hence if and only if a_0 belongs to the Jacobson radical of A.

(iv) The extension mapping sends any ideal $\mathfrak{a}$ of A to the ideal $\mathfrak{a}^e$ which consists of $f(x) = \sum a_n x^n$, $a_n \in \mathfrak{a}$. Conversely, given any ideal $\mathfrak{b}$ of $A[[x]]$, $\mathfrak{b}^c$ consists of all coefficients in any element of $\mathfrak{b}$. It's now clear that the contraction of a maximal ideal of $A[[x]]$ is maximal too and that $\mathfrak{m}^c = (\mathfrak{m}, x)$.

(v) This also follows immediately from the above.

1.6

It clearly suffices to show that every prime ideal in A is maximal. Let $\mathfrak{p}$ be a prime ideal in A and let x be a non-zero element of $A - \mathfrak{p}$. Then the ideal $(x) = \{ax/a \in A\}$ will contain an idempotent element $e \neq 0$, say $a_0 x$. This implies that $a_0 x(a_0 x - 1) = 0 \in \mathfrak{p}$, hence $a_0 x(a_0 x - 1) = 0$ in $A/\mathfrak{p}$, too. However, $A/\mathfrak{p}$ is an integral domain, therefore $e = a_0 x \neq 0$ implies $a_0 x = 1$, or that x is a unit. Hence $A/\mathfrak{p}$ is a field and this means that $\mathfrak{p}$ is maximal.

1.7

Let $\mathfrak{p}$ be a prime ideal of A. Form $A/\mathfrak{p}$, which will be an integral domain. Given any non-zero $x \in A/\mathfrak{p}$, there will be a suitable $n \in \mathbb{N} - \{1\}$, such that $x^n = x$ or equivalently $x(x^{n-1} - 1) = 0$. This implies that $x^{n-1} = 1$, hence that x is invertible. Therefore, $A/\mathfrak{p}$ is a field and thus $\mathfrak{p}$ is a maximal ideal.

1.8

Every descending chain of prime ideals has a lower bound (the intersection of them all), hence by Zorn's lemma the set $\mathrm{Spec}(A)$ has at least one minimal element (in order to apply the lemma, we order the set by $\supseteq$ rather than $\subseteq$).

1.9

Since $r(\mathfrak{a})$ is an intersection of prime ideals (those that contain $\mathfrak{a}$), we see that $r(\mathfrak{a}) = \mathfrak{a}$ implies that $\mathfrak{a}$ is an intersection of prime ideals. Conversely, if $\mathfrak{a}$ is an intersection of prime ideals, then this intersection is contained in the intersection of prime ideals of $r(\mathfrak{a})$, hence $r(\mathfrak{a}) \subset \mathfrak{a}$, which shows that $r(\mathfrak{a}) = \mathfrak{a}$ (the other direction is obvious by definition).

1.10

We have the following:

(i) $\Rightarrow$ (ii) Any maximal ideal in A (there is at least one), will be prime, hence it will coincide with the unique prime ideal $\mathfrak{a}$ of A. Hence A is a local ring and $\mathfrak{R} = \mathfrak{a}$. If we consider $A/\mathfrak{R} = A/\mathfrak{a}$ (which is a field), we deduce that every element of A is nilpotent or a unit.

(ii) $\Rightarrow$ (iii) This direction is obvious by the definition of a field.

(iii) $\Rightarrow$ (i) The nilpotent radical is maximal (and thus prime) if $A/\mathfrak{N}$ is a field. However,

$$\mathfrak{N} = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} \mathfrak{p},$$

hence $\mathfrak{N}$ is included in the intersection on the right, hence every prime ideal contains $\mathfrak{N}$. But this implies that every prime ideal coincides with $\mathfrak{N}$ (since $\mathfrak{N}$ is maximal) and so there is only one prime ideal in A.

1.11

We have the following:

(i) We just apply the given condition to $x+x$, to obtain $x+x = (x+x)^2 = x^2+x^2+2x = x+x+2x$, hence $2x = 0$.

(ii) Every prime ideal is maximal follows from exercise 7, while the second corollary follows from (i).

(iii) It suffices to show that any ideal generated by two elements of A is in fact principal. Indeed, given $a, y \in A$, we claim that $(x, y) = (x+y+xy)$. The direction $(x, y) \supset (x+y+xy)$ is trivial. For the other inclusion, note that any element of (x, y) is of the form $\sum x^m y^n$ but given the conditions of idempotency, we see that the only elements that remain after the reductions in the sum will belong to $(x+y+xy)$, hence the other direction.

1.12

If $\mathfrak{m}$ is the unique maximal ideal of the ring A, then m contains all the non-units of A. If $e \in A$ were idempotent, then $e(e-1) = 0$, hence if e or $e-1$ were units, then e would be 0 or 1. Otherwise, $e \in \mathfrak{m}$ and $e - 1 \in \mathfrak{m}$, which imply $1 \in \mathfrak{m}$, contradiction, since $\mathfrak{m}$ is by definition a proper ideal of A.

Construction of an algebraic closure of a field (E. Artin)

1.13

We will first show that $\mathfrak{a} = (\{f(x_f)\}_{f \in \Sigma})$ is not the unit ideal. Otherwise, given any polynomial $p \in A$ it would be presentable as a finite sum in the form

$$p = \sum_{f \in \Sigma} y_f f(x_f),$$

where $y_f \in A$. But 1 clearly cannot be represented in such a form, hence $\alpha \neq (1)$. If we now let $\mathfrak{m}$ be the maximal ideal of A containing $\mathfrak{a}$, we observe that $K_1 = A/\mathfrak{m}$ is a field extension of K in which every $f \in \Sigma$ has a root. Repeating the construction, we obtain K_2 in which every $f \in \Sigma$ has two roots (if possible), and similarly we obtain K_n for al $n \in \mathbb{N}$. We deduce that $L = \bigcup_{i=1}^{\infty} K_i$ is a field extension which contains all the roots of every $f \in \Sigma$; its algebraic elements form an algebraic closure $\overline{K}$ for K.

1.14

The fact that Σ has a maximal element is a trivial application of Zorn's lemma; we just need to show that every ascending chain of ideals has a maximal element. Now, assume that $\mathfrak{m}$ is a maximal ideal of Σ and let $xy \in \mathfrak{m}$, $pxy = 0$, with $p \neq 0$. We claim that $x \in \mathfrak{m}$ or $y \in \mathfrak{m}$. Assume the contrary. Then, $\mathfrak{m} \subset (\mathfrak{m}, x)$ and $(\mathfrak{m}, x)$ would still be an ideal of Σ, since its elements are clearly zero divisors. This furnishes a contradiction to the maximality of $\mathfrak{m}$. Therefore, every maximal ideal of Σ is prime.

The prime spectrum of a ring

1.15

We have the following:

(i) The relations $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$ are obvious.

(ii) Similarly, the relations $V(0) = X = \mathrm{Spec}(A)$ and $V(1) = \varnothing$ are obvious.

(iii) Again, we have

$$\bigcap_{i \in I} V(E_i) = V(\bigcap_{i \in I} E_i)$$

(iv) Similarly trivial are the relations $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.

These results show that the space $Spec(A)$ of all prime ideals of A can be endowed with a topology - the *Zariski topology* - if we define the $V(E)$ to be its closed sets.

1.16

We immediately see the following:

$\mathrm{Spec}(\mathbb{Z}) = \{(p) : p \in \mathbb{Z} \text{ is prime }\}$.

$\mathrm{Spec}(\mathbb{R}) = \varnothing$.

$\mathrm{Spec}(\mathbb{C}[x]) = \{(p) : p \in \mathbb{C}[x] \text{ is of degree } 1\}$.

$\mathrm{Spec}(\mathbb{R}[x]) = \{(p) : p \in \mathbb{R}[x] \text{ is irreducible}\}$.

$\mathrm{Spec}(\mathbb{Z}[x]) = \{(p) : p \in \mathbb{Z}[x] \text{ is irreducible}\}$.

1.17

Given $f \in A$, we define $X_f = \{\mathfrak{p} \in \mathrm{Spec}(A)/f \notin \mathfrak{p}\}$. It's obvious that $X = X_1$, $\varnothing = X_0$ and $O = X - V(E) = \bigcup_{f \in E} X_f$, hence the set $\{X_f, f \in A\}$ is a basis for the Zariski topology on $\mathrm{Spec}(A)$. We now have:

(i) $X_f \cap X_g = X_{fg}$ (obviously)

(ii) $X_f = \varnothing \Leftrightarrow f \in \mathfrak{R}$ (obviously)

(iii) $X_f = X \Leftrightarrow f \in A^\times$ (obviously)

(iv) $X_f = X_g \Leftrightarrow r(f) = r(g)$ (obviously)

(v) Note that for $f, g \in A$, $X_f = X_g$ if and only if $(f) = (g)$. In particular, $X_f = X = X_1$ if and only if $f \in A^\times$. We also easily deduce (by de Morgan's formula and exercise 15) that:

$$\bigcup_{i \in I} X_{f_i} = X_{(\{f_i\}_{i \in I})}$$

Therefore, if $\{X_{f_i}\}_{i \in I}$ is an open cover of X (and it's only those covers of X that we need to consider, by a standard proposition in point-set topology), then

$$X_{(\{f_i\}_{i \in I})} = X_1,$$

which implies that the $\{f_i\}_{i \in I}$ generate the unit ideal. Therefore, there is a *finite* subset of indices J such that

$$1 = \sum_{j \in J} g_j f_j,$$

where $g_j \in A$. Now, obviously the $\{f_j\}_{j \in J}$ generate the unit ideal, hence the $\{X_{f_j}\}_{j \in J}$ are a finite subcover of X.

(vi) This follows by exactly the same argument as above, but considering covers of the form $\{X_{f_i}\}_{i \in I}$, where $X_{f_i} \subset X_f$.

(vii) If an open subspace Y of X is quasi-compact, then considering a standard cover by sets of the form X_f, $f \in A$ we see that Y must be a finite union of sets X_f.

Conversely, if an open subspace Y of X is a union of a finite number of sets X_f, then any open cover $\{X_{f_i}\}_{i \in I}$ of Y induces an open cover for each of the X_f (namely $\{X_f \cap X_{f_i}\}_{i \in I}$). By (vii), each of those will have a finite subcover and these subcovers yield a finite subcover of X.

1.18

We have the following:

(i) By the definition of the Zariski topology, $\{x\}$ is closed in Spec(A) if and only if $\{x\} = V(E)$ for some subset E of A, hence if and only if $\mathfrak{p}_x$ is the only prime ideal that contains E, hence if and only if $E = \mathfrak{p}_x$ and $\mathfrak{p}_x$ is maximal (attaching any elements of $A - E$ would generate the unit ideal).

(ii) The relation $\overline{\{x\}} = V(\mathfrak{p}_x)$ is obvious by our remarks above.

(iii) $y \in \overline{\{x\}} = V(\mathfrak{p}_x)$ if and only if $\mathfrak{p}_y \supset \mathfrak{p}_x$

(iv) If x and y are distinct points of Spec(A), then either $\mathfrak{p}_x \not\subseteq \mathfrak{p}_y$ or $\mathfrak{p}_y \not\subseteq \mathfrak{p}_x$; assume without loss of generality the latter. This is equivalent by our previous observations to $x \notin \overline{\{y\}}$, which implies that $X - \overline{\{y\}}$ is an open set that contains x but not y.

1.19

We claim that Spec(A) is irreducible if and only $X_f \cap X_g \neq \varnothing$ for f and g non-nilpotent. Indeed, since $\{X_f\}_{f \in A}$ are a basis for the Zariski topology on Spec(A), we see that any two non-empty sets will intersect if and only if any two non-empty basis elements intersect. This is equivalent to $X_{fg} = X_f \cap X_g \neq \varnothing$ if $X_f, X_g \neq \varnothing$. The latter condition is fulfilled if and only if f and g are not nilpotent (by exercise 17) hence the previous condition is equivalent to the following: there is a prime ideal $\mathfrak{p}$ such that $fg \notin \mathfrak{p}$ if f, g are not nilpotent hence $fg \notin \mathfrak{R}$ if $f \notin \mathfrak{R}$, $g \notin \mathfrak{R}$. Therefore, X is irreducible if and only if the nilradical is prime.

1.20

We have the following:

(i) If Y is an irreducible subspace if a topological space X, then $\overline{Y}$ is also irreducible, since by definition any neighborhood of a boundary point will intersect Y (hence any two open sets in Y continue to intersect in $\overline{Y}$).

(ii) We consider the set Σ of all irreducible subspaces of X; it's not empty, since $x \in \Sigma$ for all $x \in X$. Then, by an application of Zorn's lemma in the usual fashion (any ascending chain of irreducible subspaces will be bounded by the union of all its elements which is irreducible itself) we guarantee a maximal element for Σ.

(iii) The maximal irreducible components of X are obviously closed (otherwise their irreducible closures would strictly contain them, contradiction) and they cover X (we see that any point x of X is contained in a maximal irreducible subspace by applying Zorn's lemma to the space Σ_x which comprises the irreducible subspaces of X that contain x). In a Hausdorff space each point is a maximal irreducible component.

(iv) In the case of *Spec*(A) we note that the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal are irreducible (any two open sets will be of the form $V(\mathfrak{p}) - V(E)$ and hence they will intersect) and that any two points $x \in V(\mathfrak{p}_1)$, $y \in V(\mathfrak{p}_2)$ can be separated by disjoint open sets. Therefore, the maximal irreducible components of Spec(A) are $V(\mathfrak{p})$, where $\mathfrak{p} \in$ Spec(A) is minimal.

1.21

We have the following:

(i) The following equivalences:

$$\mathfrak{q} \in \phi^{*-1}(X_f) \Longleftrightarrow \phi^*(\mathfrak{q}) \in X_f \Longleftrightarrow f \notin \phi^*(\mathfrak{q}) = \phi^{-1}(\mathfrak{q}) \Longleftrightarrow \mathfrak{q} \in Y_{\phi(f)}$$

yield that $\phi^{*-1}(X_f) = Y_{\phi(X_f)}$ Now ϕ^* is continuous, since the X_f form a basis for the Zariski topology.

(ii) The following equivalences:

$$\mathfrak{q} \in \phi^{*-1}(V(\mathfrak{a})) \Longleftrightarrow \phi^*(\mathfrak{q}) \supseteq \mathfrak{a} \Longleftrightarrow \mathfrak{q} \supseteq \mathfrak{a}^e \Longleftrightarrow \mathfrak{q} \in V(\phi(\mathfrak{a})^e)$$

yield that $\phi^{*-1}(V(\mathfrak{a})) = V(\mathfrak{a}^e)$, as desired.

(iii) The statement that

$$\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$$

follows in the same fashion as the previous one.

(iv) By proposition 1.1, we know that $\phi^*(Y) = V(\ker(\phi))$ and ϕ^* induces a bijective (and continuous, by the previous question) map from Y to $V(\ker(\phi))$. Thus we merely need to show that ϕ^{*-1} is continuous. Let $Y' = V(\mathfrak{b})$ be any closed subset of Y and let $\mathfrak{a} = \phi^{-1}(\mathfrak{b})$. Then, the following equivalences:

$$\mathfrak{p} \in \phi^*(Y') = \phi^*(V(\mathfrak{b})) \Longleftrightarrow \mathfrak{p} = \phi^*(\mathfrak{q}) \supseteq \mathfrak{b} \Longleftrightarrow \mathfrak{p} = \phi^{-1}(\mathfrak{q}) \supseteq \mathfrak{b}^c \Longleftrightarrow \mathfrak{p} \in V(\mathfrak{b}^c)$$

imply that $\phi^*(Y') = V(\mathfrak{b}^c)$ and in particular that $\phi^*(Y')$ is closed when Y' is and therefore that ϕ^* is a homeomorphism.

In particular, the natural surjective projection map from A to $A/\mathfrak{R}$ induces a homeomorphism between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A/\mathfrak{R})$.

(v) By the previous statement, we have

$$\overline{\phi^*(Y)} = \overline{\phi^*(V(0))} = V(0^c) = V(\ker(\phi))$$

thus $\phi^*(Y)$ is dense in $X \Longleftrightarrow \overline{\phi^*(Y)} = V(\ker(\phi)) = X \longleftrightarrow \ker(\phi) \subseteq \mathfrak{p}$ for all prime ideals $\mathfrak{p} \Longleftrightarrow \ker(\phi) \subseteq \mathfrak{R}$.

(vi) The desired result follows immediately by definition.

(vii) By assumption, the two only prime ideals of A are $\mathfrak{p}$ and 0, which implies that $\mathfrak{p}$ is a maximal ideal of A and thus $A/\mathfrak{p}$ is a field. This yields that the ring $B = (A/\mathfrak{p}) \times K$ will also have only two ideals, namely $\mathfrak{q}_1 = \{(\overline{x}, 0) : x \in A\}$ and $\mathfrak{q}_2 = \{(\overline{0}, k) : k \in K\}$. The ring homomorphism $\phi : A \longrightarrow B$ defined by $\phi(\overline{x}, x)$ is bijective ($\phi^*(\mathfrak{q}_1) = 0$ and $\phi^*(\mathfrak{q}_2) = \mathfrak{p}$) and continuous.

However, ϕ^* is not a homeomorphism. In the topological space *Spec*$(B) = \{\mathfrak{q}_1, \mathfrak{q}_2\}$, we have $\{\mathfrak{q}_1\} = V(\mathfrak{q}_1)$ is closed as $\mathfrak{q}_1 \nsubseteq \mathfrak{q}_2$, but $\phi^*(\mathfrak{q}_1) = 0$ is not closed in $\mathrm{Spec}(A)$, since 0 is not a maximal ideal of A (by exercise 18).

1.22

A decomposition of A in the form

$$A \simeq A_1 \times A_2 \times \ldots \times A_n$$

yields a decomposition

$$\mathrm{Spec}(A) \simeq \mathrm{Spec}(A_1) \times \mathrm{Spec}(A_2) \times \ldots \mathrm{Spec}(A_n)$$

If we embed every space $\mathrm{Spec}(A_i)$ as $X_i = (0, 0, \ldots, \mathrm{Spec}(A_i), 0, \ldots, 0)$ in $\mathrm{Spec}(A)$, it's a standard argument that the existence of the latter decomposition is equivalent to the decomposition of $\mathrm{Spec}(A)$ as a disjoint union of the X_i.

Given a ring A, we have the following:

(i) $\Rightarrow$ (ii) This direction follows from our previous observation and the definition of connectedness.

(ii) $\Rightarrow$ (iii) If a decomposition of the form $A \simeq A_1 \times A_2$ (where A_1, A_2 are non-trivial) existed, then a non-trivial idempotent element of A would be the pull-back of $(1, 0)$.

(iii) $\Rightarrow$ (ii) If $e \in A$ is a non-trivial idempotent, then $X = \mathrm{Spec}(A)$ decomposes as $X_1 \sqcup X_2$, where

$$X_1 = \{\mathfrak{p} \in X / e \in \mathfrak{p}\}$$

and

$$X_2 = \{\mathfrak{p} \in X / e - 1 \in \mathfrak{p}\}.$$

It's a trivial observation that $X_1 \cap X_2 = \varnothing$ (as in our proof that a local ring possesses no non-trivial idempotents) and similarly trivial is the verification that $X = X_1 \cup X_2$. This decomposition implies that X is disconnected.

1.23

We have the following:

(i) Each $f \in A$ is idempotent hence X_f induces a disjoint decomposition as in the previous exercise. It's now obvious that the sets X_1 and X_2 with the notation of exercise 22 are simultaneously closed and open.

(ii) By the formula of exercise 17, and by the fact that a Boolean ring is always a Principal Ideal Domain, we deduce that there is $f \in A$ such that

$$X_{(f_1, f_2, \ldots, f_n)} = X_{f_1} \cup X_{f_2} \cup \ldots \cup X_{f_n} = X_f$$

(iii) The hint in the book is a full proof; let $Y \subset X$ be both open and closed. Since Y is open, it is a union of sets X_f. Since Y is a closed subspace of a quasi-compact space, it is quasi-compact too hence it is a finite union of sets X_f, say $X_{f_1}, X_{f_2}, \ldots, X_{f_n}$. Now, (ii) finishes the proof.

(iv) X is (obviously) compact and Hausdorff.

1.24

There is nothing to be provided other than a tedious verification of the axioms.

1.25

Stone's Theorem that every Boolean lattice is isomorphic to the lattice of open and closed sets of some compact Hausdorff space follows immediately from exercises 23 and 24.

1.26

We will just repeat the construction of the book, which shows that $X \simeq \mathrm{Max}(C(X))$, by the map $\mu : X \to \mathrm{Max}(C(X))$ given by $x \mapsto \mathfrak{m}_x = \{f \in C(X)/f(x) = 0\}$. Note that $\mathfrak{m}_x$ is always a maximal ideal, as the kernel of the surjective map that sends f to $f(x)$ (whence $C(X)/\mathfrak{m}_x \simeq \mathbb{R}$ is a field).

(i) Let $\mathfrak{m}$ be any maximal ideal in $\overline{X}$. Then, let $V(\mathfrak{m})$ be the set of common zeroes of functions in $\mathfrak{m}$, namely $V(\mathfrak{m}) = \{x \in X/f(x) = 0 \text{ for all } f \in \mathfrak{m}\}$. We claim that $V(\mathfrak{m}) \neq \varnothing$. Indeed, otherwise, for every $x \in X$ there is $f_x \in \mathfrak{m}$, such that $f_x(x) \neq 0$. Since f_x is continuous, there is a neighborhood U_x of x on which f_x does not vanish. By the compactness of X, a finite number of these neighborhoods cover X, say

$$\{U_{x_i}\}_{i=1,2,\ldots n}.$$

Then, $f = \sum_{i=1}^{n} f_{x_i}^2 \in \mathfrak{m}$, but f does not vanish on any point of X, hence it's a unit, hence $\mathfrak{m} = (1)$, contradiction. Therefore, $V(\mathfrak{m}) \neq \varnothing$. Let $x \in V(\mathfrak{m})$. Then, $\subseteq \mathfrak{m}_x$, which implies $\mathfrak{m} = \mathfrak{m}_x$ by the maximality of $\mathfrak{m}$. Hence $\mathfrak{m} \in \mathrm{Im}\mu$ and μ is surjective.

(ii) By Urysohn's lemma, the continuous functions separate the points of $C(X)$ and this implies that $\mathfrak{m}_x \neq \mathfrak{m}_y$ if $x \neq y$. Hence μ is injective.

(iii) For $f \in C(X)$, let

$$U_f = \{x \in X/f(x) \neq 0\}$$

and

$$\overline{U_f} = \{\mathfrak{m} \in \overline{X}/f \in \mathfrak{m}\}.$$

We obviously have $\mu(U_f) = \overline{U_f}$. Since the open sets U_f (resp. $\overline{U_f}$) form bases of the topologies on X and $\overline{X}$ we deduce that μ is also continuous (as is μ^{-1}). Therefore X is homeomorphic to $\mathrm{Max}(C(X))$.

Affine algebraic varieties

1.27

There is nothing to be proved in this exercise if we invoke the Nullstellensatz for the surjectivity of μ.

1.28

We have the following situation:

$$\Psi : [\phi : X \to Y, \ \ \textit{regular}] \leftrightarrow \mathrm{Hom}_k(P(Y), P(X)),$$

where Ψ is defined by

$$\phi \longmapsto \Psi(\phi) : (\eta \mapsto \eta \circ \phi).$$

We see that Ψ is injective because $\eta(\phi_1) = \eta(\phi_2)$ for all η implies $\phi_1 = \phi_2$ (obviously; just let η be the natural projections). It's also surjective; if Ψ is any k-algebra homomorphism $P(Y) \to P(X)$, then $\Psi = \phi \circ \eta$, where η is the polynomial transformation that sends the values of ϕ on $P(X)$ to the values of Ψ.

Chapter 2

Modules

2.1

Since m and n are coprime, there are integers a and b such that $am + by = 1$. Therefore, given $x \otimes y \in (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$, we see that $x \otimes y = 1(x \otimes y) = am(x \otimes y) + nb(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) = a(0 \otimes y) + b(x \otimes 0) = 0$, and since every generator is identically 0, so will the whole tensor product be.

2.2

If we tensor the exact sequence

$$\mathfrak{a} \xrightarrow{incl} A \xrightarrow{\pi} A/\mathfrak{a} \longrightarrow 0,$$

we obtain

$$\mathfrak{a} \otimes_A M \xrightarrow{incl \otimes 1} A \otimes_A M \xrightarrow{\pi \otimes 1} (A/\mathfrak{a}) \otimes_A M \longrightarrow 0$$

and this induces an isomorphism between $(A/\mathfrak{a}) \otimes_A M$ and the cokernel of $incl \otimes 1$, which is $(A \otimes_A M)/(\mathfrak{a} \otimes_A M) \simeq M/\mathfrak{a}M$, since given any ideal $\mathfrak{a}$ of A a trivial argument shows that $\mathfrak{a} \otimes_A M \simeq \mathfrak{a}M$ (we need M to be flat for this to be true). Hence, $(A/\mathfrak{a}) \otimes_A M \simeq M/\mathfrak{a}M$, as desired.

The above proposition is true even if M is not a flat A-module. A proof in this general case would proceed as follows: consider the map $\phi : (A/\mathfrak{a}) \otimes_A M \longrightarrow M/\mathfrak{a}M$, defined by $(a, x) \mapsto ax \bmod \mathfrak{a}M$. This is clearly a bilinear homomorphism, which induces a linear homomorphism $M \longrightarrow (A/\mathfrak{a}) \otimes_A M$ whose inverse is $x \mapsto \overline{1} \otimes_A x$ (where $\overline{1}$ is the image of 1 in $A/\mathfrak{a}$). It is clear that $\mathfrak{a}M$ is contained in the kernel of this last linear map, and hence the construction yields an isomorphism between $M/\mathfrak{a}M$ and $(A/\mathfrak{a}) \otimes_A M$, as desired (Lang, *Algebra*, 612).

2.3

Let $\mathfrak{m}$ be the maximal ideal of A, $k = A/\mathfrak{m}A$ its residue field. We also let $M_k = k \otimes_A M \simeq M/\mathfrak{m}M$, by exercise 2. The condition $M \otimes_A N = 0$ implies $(M \otimes_A N)_k = 0$, hence $M_k \otimes_A N_k = (M \otimes_A N)_k \otimes_A k = 0$. But M_k and N_k are vector fields over the field k, hence $\dim_k(M_k \otimes_A N_k) = \dim_k(M_k)\dim_k(N_k)$, hence $M_k \otimes_A N_k = 0$ implies $M_k = 0$ or $N_k = 0$. Let without loss of generality the former be true. Then, since $\mathfrak{m}$ is the unique maximal ideal, it will coincide with the Jacobson radical of A, so $M/\mathfrak{m}M = M_k = 0$ implies $M = \mathfrak{m}M$ and by Nakayama's lemma this yields $M = 0$.

2.4

If M, N, P are A-modules, we know that

$$(M \oplus N) \otimes P = (M \otimes P) \oplus (N \otimes P),$$

hence we have the following equivalence: $M = \bigoplus_{i \in I} M_i$ is flat if and only if an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

remains exact after tensoring by M:

$$0 \longrightarrow M' \otimes (\bigoplus_{i \in I} M_i) \longrightarrow M \otimes (\bigoplus_{i \in I} M_i) \longrightarrow M'' \otimes (\bigoplus_{i \in I} M_i) \longrightarrow 0.$$

The above can be written as

$$0 \longrightarrow \bigoplus_{i \in I} (M' \otimes M_i) \longrightarrow \bigoplus_{i \in I} (M \otimes M_i) \longrightarrow \bigoplus_{i \in I} (M'' \otimes M_i) \longrightarrow 0,$$

and this sequence is exact if and only if each component

$$0 \longrightarrow (M' \otimes M_i) \longrightarrow (M \otimes M_i) \longrightarrow (M'' \otimes M_i) \longrightarrow 0$$

is exact, hence if and only if each M_i is flat.

2.5

We observe that

$$A[x] = \bigoplus_{m=0}^{\infty} (x^m),$$

hence $A[x]$ is a flat A-algebra if and only if each component (x^m) is a flat A-algebra (by the previous exercise). By Lang's lemma (Lang, *Algebra*, 618), it suffices to show that the natural map $\phi : \mathfrak{a} \otimes (x^m) \longrightarrow \mathfrak{a}(x^m)$ is an isomorphism for any ideal $\mathfrak{a}$ of A. Indeed, surjectivity is obvious and we may write any arbitrary generator of $\mathfrak{a} \otimes (x^m)$ as $a \otimes x^m$ (transferring the constants to the first slot; the product will be in $\mathfrak{a}$, because $\mathfrak{a}$ is an ideal). Any two such generators will map to the same element in $\mathfrak{a}(x^m)$ is and only if their respective first slots coincide and thus if and only if they coincide. Hence the natural map is an isomorphism, as desired.

2.6

First note that $M[x] = \{m_0 + m_1 x + \ldots + m_r x^r / m_i \in M, r \in \mathbb{N}\}$ is a module over $A[x]$, if we define the product of two polynomials in the obvious fashion. We see that since $M = AM = \{am/a \in A, m \in M\}$, $M[x]$ will coincide with $(AM)[x]$.

We also see that the homomorphism $\phi : A[x] \otimes_A M \longrightarrow (AM)[x]$ defined as $a(x) \otimes_A m \mapsto a(x)m$ has an obvious inverse $\psi : a(x)m \mapsto a(x) \otimes_A m$ and thus it induces an isomorphism of tensor products between $A[x] \otimes_A M$ and $(AM)[x] = M[x]$, as desired.

2.7

Note that if A is a ring and $\mathfrak{a}$ is any ideal in it, then

$$A[x]/\mathfrak{a}[x] \simeq (A/\mathfrak{a})[x].$$

For the proof, just note that $\mathfrak{a}[x]$ is the kernel of the natural projection map from $A[x]$ to $(A/\mathfrak{a})[x]$.

Now let $\mathfrak{p}$ be a prime ideal of A. Then, $A/\mathfrak{p}$ is an integral domain, and hence so is $(A/\mathfrak{p})[x]$ by the Hilbert Basis Theorem. But, by the above, $A[x]/\mathfrak{p}[x]$ will be an integral domain too, thus $\mathfrak{p}[x]$ sill be a prime ideal in $A[x]$, as desired.

In the case $\mathfrak{p}$ is maximal in A, it doesn't necessarily follow that $\mathfrak{p}[x]$ is maximal in $A[x]$; if $\mathbb{F}$ is a field, it doesn't necessarily follow that $\mathbb{F}[x]$ is a field too.

2.8

We have the following:

(i) If M and N are flat A-modules, then so is $M \otimes_A N$, since the tensor functor is associative (tensoring an exact sequence first by M and then by N is equivalent to tensoring by $M \otimes_A N$).

(ii) Note first that if

$$0 \longrightarrow M' \xrightarrow{1\otimes f} B \otimes_A M \xrightarrow{1\otimes g} B \otimes_A M'' \longrightarrow 0$$

is an exact sequence and B is flat, then

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is also exact. By proposition 2.19, this boils down to the statement that if the homomorphism $1 \otimes_A f$ is injective, then so is the homomorphism f. For assume that $1 \otimes_A f$ is injective, but f isn't. Then, there are distinct x_1, x_2 such that $f(x_1) = f(x_2)$, and for every suitable y we would have $y \otimes_A f(x_1) = y \otimes_A f(x_2)$ hence $y \otimes_A x_1 = y \otimes_A x_2$ (by the injectivity of $1 \otimes_A f$). But then, $(1) \otimes_A (x) = 0$ (where $x = x_1 - x_2$) and since both modules are finitely generated, we deduce (by exercise 3) that $x = 0$, contradiction.

Now if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

is exact, then by the assumptions of the exercise it will remain exact when tensoring first by B (after which we may regard the sequence as a sequence of B-modules) and then by N. The above lemma implies that we may remove the tensor by the flat A-module B without penalty and this will leave us with an exact sequence; but that's merely the initial sequence tensored by N. Hence N is flat as an A-module, as desired.

2.9

We shall only use the assumption that $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact and M'' is finitely generated. We see that if $x_1, x_2, \ldots, x_n$ are generators for M'', then $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ in $\mathrm{Coker}(g) = M''/f(M')$ will generate $\mathrm{Coker}(g)$. But by the exactness of the sequence, $\mathrm{Coker}(g) \simeq M$, hence M will be finitely generated.

2.10

We shall first show the following embedding:

$$(N/u(M))/(\mathfrak{a}(N/u(M))) \hookrightarrow (N/\mathfrak{a}N)/\overline{u}(M/\mathfrak{a}M)$$

The mapping is the natural one: given $\overline{n} \in N/u(M)$ we send it first to $\tilde{n} \in N/\mathfrak{a}N$ and then to $\hat{\tilde{n}} \in (N/\mathfrak{a}N)/\overline{u}(M/\mathfrak{a}M)$. It's a trivial verification that the kernel of this A-module homomorphism is included in $\mathfrak{a}(N/u(M))$, hence the embedding. However, we notice that $(N/\mathfrak{a}N)/\overline{u}(M/\mathfrak{a}M) = 0$ since the induced homomorphism is surjective. Therefore, we will have $(N/u(M))/(\mathfrak{a}(N/u(M))) = 0$, too and by Nakayama's lemma $N/u(M) = 0$, hence $N = u(M)$, hence u is surjective, as desired.

2.11

We have the following:

(i) Let $\phi : A^n \longrightarrow A^m$ be an isomorphism and let $\mathfrak{m}$ be a maximal ideal of A. Then, $\mathfrak{m}$ annihilates the module $(A/\mathfrak{m}) \otimes_A A^m$, hence we may regard $(A/\mathfrak{m}) \otimes_A A^m$ (as well as $(A/\mathfrak{m}) \otimes_A A^n$, of course) as a vector space over the field $A/\mathfrak{m}$. But then ϕ induces an isomorphism

$$1 \otimes \phi : (A/\mathfrak{m}) \otimes_A A^m \longrightarrow (A/\mathfrak{m}) \otimes_A A^n$$

between two vector spaces of dimensions n and m and this clearly implies $m = n$.

(ii) If $\phi : A^m \longrightarrow A^n$ is surjective, then $A^n \simeq A^m/N$, where $N = \ker\phi$. But, if $x_1, x_2, \ldots x_m$ generate A^m, then obviously $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m$ generate A^n, hence $m \geq n$, as desired.

(iii) Thanks to Nick Rozenblyum for informing me that the statement is in fact correct, but it's probably one of the most difficult problems in the book!

If $\phi : A^m \longrightarrow A^n$ is injective, then it **necessarily** follows that $m \leq n$. Indeed, let $\{e_1, e_2, \ldots, e_m\}$ be the standard canonical basis of A^m and let $\phi(e_i) = (a_{i1}, a_{i2}, \ldots, a_{in}) \in A^n$ for $1 \leq i \leq m$. Then, let D be the $n \times n$ matrix (a_{ij}). Without loss of generality, we may consider the cases $m, n > 0$ and $D \neq 0$ (the omitted cases are trivial); also, by possibly rearranging the orders of the basis elements of A^m and A^n, we may assume that the non-zero $r \times r$ minor of D is at the upper left corner.

Suppose, contrary to the desired conclusion, that $m > n$. Then, $m \geq r + 1$ and we denote the $(r+1) \times r$ block matrix at the upper left corner of D by D'. For each $j = 1, 2, \ldots, n$ the quantity $\sum_{i=1}^{r+1} a_{ij} b_i$ can be realized as the determinant of a $(r+1) \times (r+1)$ matrix. If $1 \leq j \leq r$, then the matrix has two identical columns up to ± 1. If $r + 1 \leq j \leq n$, then the determinant of the matrix is an $(r+1) \times (r+1)$ minor (again, up to ± 1) of D. Therefore, we have $\sum_{i=1}^{r+1} a_{ij} b_i = 0$ for all $1 \leq j \leq n$. But this means that

$$\phi(b_1, \ldots, b_{r+1}, 0, \ldots, 0) = (\sum_{i=1}^{r+1} a_{i1} b_i, \ldots, \sum_{i=1}^{r+1} a_{in} b_i) = (0, \ldots, 0) \in A^n$$

which is a contradiction to the injectivity of ϕ since $(b_1, \ldots, b_{r+1}, \ldots, 0, \ldots, 0) \neq (0, 0, \ldots, 0) \in A^m$.

This completes the proof that $m \leq n$.

2.12

Let $e_1, e_2, \ldots, e_n$ be a basis of A^n. Choose $u_1, u_2, \ldots, u_n$ such that $\phi(u_i) = e_i$. We now claim that $M = \ker \phi \oplus (u_1, u_2, \ldots, u_n)$. Indeed, if $x \in M$, then $\phi(x) = \sum_{i=1}^n y_i e_i = \phi(\sum_{i=1}^n y_i u_i)$ for some $y_i \in A$, hence there is a unique $n \in \ker \phi$ such that $x = n + \sum_{i=1}^n y_i u_i$, which clearly implies $M = \ker \phi \oplus (u_1, u_2, \ldots, u_n)$. Since there is a finite number of generators $x_1, x_2, \ldots, x_m$ of M, we see that at most a finite number of them generate $(u_1, u_2, \ldots, u_n)$ (we may add the u_i to the x_i if necessary) and hence the complement of $(u_1, u_2, \ldots, u_n)$ in M, namely $\ker \phi$, will be generated by the rest of the x_i. In particular, it will be finitely generated.

2.13

In order to show the injectivity of g, we will repeat a familiar argument: let y map to 0 under g. Then, $1 \otimes y = 0$, thus $(1) \otimes (y) = 0$, hence $(y) = 0$ by exercise 3, since (1) and (y) are both finitely generated. This yields $y = 0$, as desired.

Then, let $p : N_B \longrightarrow N$ be defined by sending $b \otimes y$ to by. We now claim that $\ker p$ and $g(N)$ are direct summands of N_B and moreover $N_B = g(N) \oplus \ker p$. Indeed, we obviously have $g(N) \cap \ker p = 0$, and also $N_B = g(N) + \ker p$, since any *generator* $b \otimes n$ of N_B can be written as $b(1 \otimes n) + 0 \otimes n$. This completes the proof.

Direct limits

2.14

We will just repeat the construction of the book; there is nothing else to be proved. Let A be a ring, I a directed set and let $(M_i)_{i \in I}$ be a family of A-modules indexed by I. For each pair $i, j \in I$ such that $i \leq j$, let $\mu_{ij} : M_i \to M_j$ be a homomorphism and suppose that the following axioms are satisfied:

(i) μ_{ii} is the identity mapping on M_i.

(ii) $\mu_{ij} = \mu_{kj} \mu_{ik}$, for elements $i \leq j \leq k$ of I. Then the modules M_i and homomorphisms μ_{ij} are said to form a *direct system* $M = (M_i, \mu_{ij})$ over the directed set I.

We shall construct an A-module M called the *direct limit* of the direct system M. Let C be the direct sum of the M_i; identify each module M_i with its embedding in C. Then let D denote the submodule generated by all elements of the form $x_i - \mu_{ij}(x_i)$ for $i \leq j$ and $x_i \in M_i$. Let M be C/D, let $\mu : C \to M$ be the natural projection and let μ_i be the restriction of μ on M_i.

The module M together with the family of homomorphisms $\mu_i : M_i \to M$, is called the *direct limit* of the direct system M. It's denoted by

$$\varinjlim M_i.$$

From the construction, it is clear that $\mu_i = \mu_j \circ \mu_{ij}$, for $i \leq j$.

2.15

Any element of $M = C/D$ is in the image of the natural projection map $\mu : C \to C/D$. Hence, any element of M can be written as $\mu(x)$, where $x \in M_i$ for some $i \in I$, since M is the direct sum of the M_i. But $\mu_i \equiv \mu|_{M_i}$, hence any element can be written as $\mu_i(x_i)$, for some $i \in I$.

Also, if an element $\mu_i(x_i)$ is equal to zero, then x_i must belong to the ideal generated by some element $x_i' - \mu_{ij}(x_i')$ for some $i \leq j$. But then, $\mu_{ij}(x_i) \in (\mu_{ij}(x_i') - \mu_{ij}(x_i')) = (0)$, hence $\mu_{ij}(x_i) = 0$ as desired.

2.16

In the situation of exercise 14, we will show that if M' were any A-module such that given any A-module N and a collection of homomorphisms $\alpha_i : M_i \longrightarrow N$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ for all $i \leq j$, then there exists a unique homomorphism $\alpha : M' \longrightarrow N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$, then M' is isomorphic to M. Note of course that M has the property itself; we just define $\alpha(x) = a_i(x_i)$, where $x = \mu_i(x_i)$ (as in the previous exercise).

If the above property is shared by some module M', then taking $N = M$ and $\alpha_{ij} = \mu_{ij}$ yields a unique homomorphism $\alpha : M' \longrightarrow M$ such that $\mu_i = \alpha \circ \mu_i$ for all $i \in I$. But then α is surjective and injective, hence an isomorphism.

2.17

By the construction, it is clear that in this case

$$\varinjlim M_i = \sum_{i \in I} M_i = \bigcup_{i \in I} M_i,$$

the last equality being an equality of sets. The first equality is true because modding out by D has the effect of identifying the canonical images of the x_i thus cancelling the distinction between elements whose *sets* of coordinates coincide (they are different in $\bigoplus M_i$ but not in $\sum M_i$).

In particular, we deduce that any A-module is the direct limit of its finitely generated submodules.

2.18

Indeed, Φ defines a unique map

$$\phi = \varinjlim : M \longrightarrow N.$$

Given $x \in M$ there is $x_i \in M_i$ for some $i \in I$ such that $\mu_i(x_i) = x$. We define $\phi(x)$ as:

$$\phi(x) = \nu_i(\phi_i(x_i)),$$

so that the required condition is satisfied. We see that ϕ is well defined because if $x = \mu_i(x_i) = \mu_i(x_i')$, then $\nu_i(\phi_i(x_i)) = \nu_i(\phi_i(x_i'))$, since by the given condition and exercise 14, $\nu_i \circ \phi_i = \nu_j \circ \nu_{ij} \circ \phi_i = \nu_j \circ \phi_j \circ \mu_{ij}$.

2.19

We will prove that if (M_i, μ_i), (N_i, ν_i) and (P_i, ρ_i) are direct systems over a common index set I such that there are families $\{\phi\}_{i \in I}$ and $\{\psi\}_{i \in I}$ of A-module homomorphisms that render the sequence

$$M_i \xrightarrow{\phi_i} N_i \xrightarrow{\psi_i} P_i$$

exact for all $i \in I$, then the corresponding sequence

$$M \xrightarrow{\phi} N \xrightarrow{\psi} P$$

we obtain after passing to the limit is also exact.

For every $i \in I$ we have $\mathrm{Im}\phi_i = \ker \psi_i$. Now, given $x \in \ker \psi$ we see by exercise 18 that, for some $i \in I$, $x \in \mathrm{Im}\phi_i \subset \mathrm{Im}\phi$, hence $\mathrm{Im}\phi \supset \ker \psi$. Similarly the other direction shows $\mathrm{Im}\phi = \ker \psi$, which is equivalent to exactness of the desired sequence.

Tensor products commute with direct limits

2.20

We will show that $\psi : P \longrightarrow M \otimes N$ is an isomorphism, so that

$$\varinjlim (M_i \otimes N) \simeq (\varinjlim M_i) \otimes N.$$

To achieve that, we just need to exhibit an inverse for ψ. This inverse will be the map $\phi : M \otimes N \longrightarrow P$ that arises from $g : M \times N \longrightarrow P$ which in turn is defined by the canonical mapping $g_i : M_i \times N \longrightarrow M_i \otimes N$ for every $i \in I$. The uniqueness of g is guaranteed by exercise 16. We further see that $\phi \circ \psi$ and $\psi \circ \phi$ are the identity, hence they are isomorphisms, as desired.

2.21

In order to show that the mappings $\alpha_i : A_i \longrightarrow A$ are ring homomorphisms we need to show they map 1_{A_i} to 1_A (the other conditions are already satisfied). But this is obvious.

If $A = 0$, then $1 = 0$, hence there is $i \in I$ such that $\alpha_i(x_i) = 1 = 0$ and hence $x_i = 0$. But $x_i = 1$, too. Thus $A_i = 0$.

2.22

It's obvious that the nilradical of the direct limit of the A_i is the direct limit of their nilradicals $\mathfrak{R}_i$ and that if each A_i is an integral domain then so is their direct limit.

2.23

There is nothing to be added to the construction of the book.

Flatness and Tor

2.24

We have the following:

(i) $\Rightarrow$ (ii) Let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence. Then, since M is flat, tensoring by M will leave the sequence exact, hence its homology groups, which are precisely $\mathrm{Tor}_n^A(M, N)$, will be 0 for all $n > 0$ and all A-modules N ($N = N, N', N''$ if we take the above exact sequence).

(ii) $\Rightarrow$ (iii) This follows directly.

(iii) $\Rightarrow$ (i) As before, let $0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$ be an exact sequence. Then, the Tor sequence

$$\mathrm{Tor}_1(M, N'') \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes N'' \longrightarrow 0$$

will be exact. But the condition $\mathrm{Tor}_1^A(M, N'') = 0$ yields that the initial sequence tensored by M is exact, or that M is flat.

2.25

Let $0 \longrightarrow E' \longrightarrow E$ be an exact sequence. Then we have the following exact and commutative diagram:

$$\begin{tikzcd}
0 \arrow[rrr] & & & N'' \otimes E' & \\
0 \arrow[r] & N' \otimes E' \arrow[r] \arrow[d] & N \otimes E' \arrow[r] \arrow[d] & N'' \otimes E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N' \otimes E \arrow[r] & N \otimes E \arrow[r] & N'' \otimes E \arrow[r] & 0
\end{tikzcd}$$

In it, the 0 on top is justified by the hypothesis that N'' is flat, and the two zeroes on the left are justified by the Tor exact sequence of homological algebra. If N' is flat, then the first vertical map is an injection , and the snake lemma shows that N is flat. Conversely, if N is flat, then the middle column is an injection. The two zeroes on the left and the commutativity of the left square show that the map $N' \otimes E' \longrightarrow N' \otimes E$ is an injection, so N' is flat. This completes the proof. (Lang, *Algebra*, 616)

2.26

First, assume that $\mathrm{Tor}_1(M, N) = 0$ for all finitely generated modules M of A. Since any A-module is a direct limit of its finitely generated submodules, and taking direct limits is an *exact operation* (by exercises 17 and 19 respectively), we obtain that $\mathrm{Tor}_1(M, N) = 0$ for all A-modules M, hence N is flat. We can reduce this condition to $\mathrm{Tor}_1(M, N) = 0$ for all *cyclic* modules M over A. For if M is finitely generated by $x_1, x_2, \ldots, x_m$ over A, and $M' = (x_m)$, $M'' = (x_1, x_2, \ldots, x_{m-1})$, then $M = M' \oplus M''$ and N is flat for M if and only if it's flat for M' and M'' (by exercise 4); but both of those have less generators than M, so the condition can be relaxed to cyclic A-modules (in particular, modules of the form $A/\mathfrak{a}$ for some ideal $\mathfrak{a}$ of A). By proposition 2.19, we may just consider finitely generated ideals $\mathfrak{a}$.

The other direction is obvious.

2.27

We have the following:

(i) $\Rightarrow$ (ii). Let $x \in A$. Then $A/(x)$ is a flat A-module, hence in the diagram

$$\begin{tikzcd}
(x) \otimes A \arrow[r, "\beta = 1 \times \pi"] \arrow[d] & (x) \otimes A/(x) \arrow[d, "\tilde{\pi}"] \\
A \arrow[r] & A/(x)
\end{tikzcd}$$

the mapping α is injective. Hence $\mathrm{Im}(\beta) = 0$, hence $(x)^2 = (x^2) = (x)$

(ii) $\Rightarrow$ (iii) Let $x \in A$. Then, by the assumption $x = ax^2$ for some $a \in A$, hence $e = ax$ is idempotent and $(e) = (x)$. Now, if e and f are idempotents, then $(e, f) = (e + f + ef)$ (as in exercise 1.11). This implies that every finitely generated ideal has an idempotent generator (in particular, it's principal) and it's will also be a direct summand since $A = (e) \oplus (e - 1)$ (as in exercise 1.22).

(iii) $\Rightarrow$ (i) Let N be any A-module. Then, given any finitely generated ideal $\mathfrak{a}$ of A, there is a **finitely generated** (by the proof of the previous direction) ideal $\mathfrak{b}$ such that $A = \mathfrak{a} \oplus \mathfrak{b}$. But this implies that $\mathrm{Tor}_1(A/\mathfrak{a}, N) = \mathrm{Tor}_1(\mathfrak{b}, N) = 0$, which is equivalent to flatness (by exercise 26).

2.28

Any ring A for which given any $x \in A$ there is $n \in \mathbb{N}$ such that $x^n = x$ is absolutely flat. This is so because $(x) = (x^{n-1})$ and x^{n-1} is idempotent, hence so is (x). By exercise 27, this condition is equivalent to absolute flatness. A Boolean ring is merely a special case of the above.

Let $f : A \twoheadrightarrow B$ be a surjective homomorphism, so that $B = f(A)$ and B is isomorphic to A/N, where $N = \ker(f)$. Then, the principal ideals of A/N are in bijective correspondence with the principal ideals of A that contain N. Any such ideal $\mathfrak{a}$ gives rise to a decomposition $A = \mathfrak{a} \oplus \mathfrak{b}$, hence $B \simeq A/N = (\mathfrak{a}/N) \oplus (\mathfrak{b}/N)$. Hence every principal ideal of B is a direct summand of B. By exercise 27, this condition is equivalent to absolute flatness.

Let $x \in A$ be an element of an absolutely flat local ring A (whose maximal ideal is $\mathfrak{m}$). Then, if x is a non-zero element of A, we have $x(ax - 1) = 0$ for some $a \in A$. If $x \notin \mathfrak{m}$, then x is a unit. If $x \in \mathfrak{m}$, then necessarily $ax - 1 \notin \mathfrak{m}$ for otherwise $1 = ax - (ax - 1) \in \mathfrak{m}$, contradiction. Therefore, $ax - 1$ is a unit in this case, but then the above equation yields $x = 0$, contradiction. Therefore, x is a unit and since x was arbitrary, we conclude that A is a field.

If A is absolutely flat and $x \in A$ is a non-unit, then $x = ax^2$ for some $a \in A$, hence $x(ax - 1) = 0$ and $ax - 1 \neq 0$, hence x is a zero divisor.

Chapter 3

Rings and Modules of Fractions

3.1

If there is $s \in S$ such that $sM = 0$, then obviously $S^{-1}M = 0$ (because then $m/t = ms/ts = 0/ts = 0$ for any elements $m/t \in S^{-1}M$). Conversely, if $S^{-1}M = 0$ and $x_1, x_2, \ldots, x_m$ generate M, then there are elements of S $s_1, s_2, \ldots, s_m$ such that $s_i m_i = 0$ for all $1 \leq i \leq n$. We may then put $s = s_1 s_2 \ldots s_m$ and this choice will clearly do.

3.2

Let a/s, $a \in \mathfrak{a}$, $s \in 1 + \mathfrak{a}$ be an element of $S^{-1}\mathfrak{a}$ and a'/t, $a' \in A$, $t \in 1 + \mathfrak{a}$ be an element of $S^{-1}A$. Then, we will show that $1 + aa'/st \in (S^{-1}A)^\times$, hence that $S^{-1}\mathfrak{a} \subset \mathfrak{J}$ (the Jacobson radical of $S^{-1}A$). Indeed, $1 + aa'/st = (st + aa')/st$ whose inverse $st/(st + aa')$ belongs to $S^{-1}A$, since $st + aa' \equiv 1 \bmod \mathfrak{a}$.

We may use this fact to show (2.5) without resorting to determinants. For if M is finitely generated, then so is $S^{-1}M$ and hence $M = \mathfrak{a}M$ implies $S^{-1}M = (S^{-1}\mathfrak{a})(S^{-1}M)$, hence Nakayama implies that $S^{-1}M = 0$. By exercise 1, there is $s \in 1 + \mathfrak{a}$ such that $sM = 0$, as desired.

3.3

We let U mean $f(T)$, where $f : T \longrightarrow S^{-1}A$ maps $t \in T$ to $t/1$, and $ST = \{st/s \in S, t \in T\}$ (so that ST is multiplicative if so are S and T). Under these assumptions, the homomorphism $\phi : (ST)^{-1}A \longrightarrow U^{-1}(S^{-1}A)$, defined as $\phi(a/st) = (a/s)/(t/1)$ is an isomorphism (obviously).

3.4

The $S^{-1}A$-module homomorphism $\phi : S^{-1}B \longrightarrow T^{-1}B$ defined by $b/s \mapsto b/f(s)$ has an obvious inverse (the map that sends $b/f(s)$ to b/s) and therefore it's an isomorphism.

3.5

Suppose that A has a non-zero nilpotent element x. Then, x belongs to all prime ideals $\mathfrak{p}$ of A and so do all its powers $\{x^n\}_{n \in \mathbb{N}}$; thus given any prime ideal $\mathfrak{p}$, the element $(x/1) \in A_\mathfrak{p}$ is nilpotent. If we now choose the maximal (thus prime) ideal $\mathfrak{p}$ that contains the ideal $\mathrm{Ann}(x)$, we make sure that $x \neq 0$ in $A_\mathfrak{p}$, hence if $A_\mathfrak{p}$ contains no non-zero nilpotent element for all prime ideals $\mathfrak{p}$, then A contains no non-zero nilpotent element.

The analogous statement for integral domains is not true though. For example, $A = \mathbb{Z}/6\mathbb{Z}$ is not an integral domain, but $A_\mathfrak{p}$ is an integral domain for all prime ideals of A (namely, (2) and (3)).

3.6

The fact that $\Sigma = \{S \subseteq A : S$ is multiplicatively closed and $0 \notin S\}$ has maximal elements follows from a trivial application of Zorn's lemma (given an ascending chain, we consider the union of all its elements); it's also trivial that if $S \in \Sigma$ is maximal, then $A - S$ will be a minimal prime **subset** of A.

In order to show that $A - S$ is an ideal, we need to show that it's closed under addition and scalar multiplication. Indeed, if S is maximal, then we have $a \notin S$ if and only if there is $s \in S$ and $n \in \mathbb{N}$ such that $sa^n = 0$. For the proof of this, observe that the set $a^{\mathbb{Z}_{\geq 0}}S = \{a^n s : n \geq 0, s \in S\}$ is multiplicatively closed and contains S and a. Now, if $a \notin S$, $b \notin S$ and $a^n s_1 = b^m s_2 = 0$, then $(ab)^n s_1 = 0$ and $(a+b)^{m+n} s_1 s_2 = 0$, hence S is closed under addition and multiplication, as desired.

Conversely, let $\mathfrak{p}$ be a minimal prime ideal of A. Then, $A - \mathfrak{p}$ is obviously in Σ and any $S' \in \Sigma$ that contained S would yield a prime ideal $\mathfrak{p}' = A - S'$ that would be contained in $\mathfrak{p}$, contradiction.

3.7

We have the following:

(i) We shall prove that if S is saturated, then

$$A - S = \bigcup_{\mathfrak{p} \cap S = \varnothing} \mathfrak{p}.$$

Indeed, it's obvious that the left hand side contains the right hand side. For the other direction, let $x \in A - S$ and let $(x) = \mathfrak{a}$. Then, if we let $S/\mathfrak{a} = \{s + \mathfrak{a} : s \in S\}$ be the set of cosets of S in $\mathfrak{a}$, we observe that $\overline{0} = 0 + \mathfrak{a} \notin S/\mathfrak{a}$ (because that would imply $S \cap \mathfrak{a} \neq \varnothing$ which is absurd by the saturation condition on S). Also, $S/\mathfrak{a}$ is multiplicatively closed as a subset of $A/\mathfrak{a}$ because S is. Therefore, by exercise 6 there exists a maximal multiplicatively closed set $\Sigma = \mathfrak{a} - \overline{\mathfrak{p}}$, where $\overline{\mathfrak{p}}$ is a minimal prime ideal such that $S \subseteq \Sigma$. This $\overline{\mathfrak{p}}$ must be of the form $\mathfrak{p}/\mathfrak{a}$ where $\mathfrak{p}$ is some prime ideal that contains $\mathfrak{a}$; in particular, $\mathfrak{p} \cap S = \varnothing$ and this shows that x belongs to the right hand side. Thus the other inclusion

$$A - S \subseteq \bigcup_{\mathfrak{p} \cap S \varnothing} \mathfrak{p},$$

holds and this completes the proof.

(ii) If S is multiplicatively closed, then

$$\overline{S} = A - \bigcup_{\mathfrak{p} \cap S = \varnothing} \mathfrak{p}$$

is a saturated and multiplicatively closed set that contains it. If $\overline{S}'$ were any saturated and multiplicatively closed subset of $\overline{S}$ that contained S, then its complement in A would contain at least one prime ideal that has non-trivial intersection with S. But then, this intersection would belong to both S and to $A - \overline{S}' \subseteq A - S$, a contradiction. Therefore S is minimal.

Now if we put $S = 1 + \mathfrak{a}$, we immediately see that $\mathfrak{p} \cap (1 + \mathfrak{a}) \neq \varnothing$ if and only if $\mathfrak{p} + \mathfrak{a} = (1)$. Therefore, $\mathfrak{p} \cap (1 + \mathfrak{a}) = \varnothing$ if and only if there is some prime ideal $\mathfrak{p}'$ such that $\mathfrak{p}' \supseteq \mathfrak{p} + \mathfrak{a}$. But then

$$\overline{1 + \mathfrak{a}} = A - \bigcup_{\mathfrak{p} \cap (1+\mathfrak{a}) = \varnothing} \mathfrak{p} = A - \bigcup_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}.$$

3.8

We have the following:

(i) $\Rightarrow$ (ii) In particular, ϕ is surjective and this implies that $t/1 \in T^{-1}A$ can be represented as a/s for some $a \in A$, $s \in S$. Now, if $a \in S$, then $t/1 = a/s$ is invertible in $S^{-1}A$. Otherwise, it's the relation $(st - a)u = 0$ for some $u \in S$ that implies invertibility of $t/1$.

(ii) $\Rightarrow$ (iii) If the inverse of $t/1$ is x/y, then there is $u \in S$ such that $(ux)t = yu \in S$, as desired.

(iii) $\Rightarrow$ (iv) It's obvious that no element of T could belong to a prime ideal that meets S, therefore

$$T \subseteq \bigcup_{\mathfrak{p} \cap S = \varnothing} \mathfrak{p} = \overline{S},$$

as desired.

(iv) $\Rightarrow$ (v) This is an immediate corollary of the above.

(v) $\Rightarrow$ (i) We will show that the induced map

$$\phi_{\overline{\mathfrak{p}}} : (S^{-1}A)_{\overline{\mathfrak{p}}} \longrightarrow (T^{-1}A)_{\overline{\mathfrak{p}}}$$

is bijective for all prime ideals $\overline{\mathfrak{p}}$ of $S^{-1}A$ and this will conclude the proof by proposition 3.9 in the book. Note that any such ideal must be of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A that doesn't meet S; by (iv) this condition yields $\mathfrak{p} \cap T = \varnothing$ too.

Indeed, an element of $(S^{-1}A)_{S^{-1}\mathfrak{p}}$ has the form $\frac{a/s_1}{f/s_2}$, where $a \in A$, $f \in A - \mathfrak{p}$ and $s_1, s_2 \in S$. If

$$\phi_{S^{-1}\mathfrak{p}} = \frac{a/s_1}{f/s_2} = 0,$$

then there would be $f'/s_3 \in S^{-1}\mathfrak{p}$ such that

$$\frac{f'}{s_3}\frac{a}{s_1} = \frac{f'a}{s_3 s_1} = 0$$

This in turn implies that there exists $t \in T$ such that $tf'a = 0$ in A, But then $tf' \notin \mathfrak{p}$ since $f' \notin \mathfrak{p}$ and $T \cap \mathfrak{p} = \varnothing$, thus $tf'/1 \notin S^{-1}\mathfrak{p}$. Hence

$$\frac{a/s_1}{f/s_2} = \frac{tf'a/s_1}{tf'f/s_2} = 0$$

and $\phi_{S^{-1}\mathfrak{p}}$ is injective, as desired.

In order to show surjectivity, note that an element of $(T^{-1}A)_{S^{-1}\mathfrak{p}}$ is of the form $\frac{a/t}{f/s}$, where $a \in A$, $s \in S$, $f \in A - \mathfrak{p}$ and $t \in T$. Since $t \in \mathfrak{p}$, we have $\frac{1}{t/1} \in (T^{-1}A)_{S^{-1}\mathfrak{p}}$. Thus

$$\frac{a/t}{f/s} = \phi_{S^{-1}\mathfrak{p}}(\frac{ta/1}{tf/s})$$

and $\phi_{S^{-1}\mathfrak{p}}$ is surjective, as desired. This completes the proof.

3.9

Note first that the set S_0 of all non-zero divisors of A is a saturated multiplicatively closed subset of A, whence it follows that D is a union of prime ideals. A prime ideal $\mathfrak{p}$ of A is minimal if and only if $S = A - \mathfrak{p}$ is a maximal multiplicatively closed subset of A that doesn't contain 0; thus it follows that S the complement in A of any union of prime ideals; in particular, $A - \mathfrak{p} = S \supseteq A - D$ and so $\mathfrak{p} \subseteq D$. We then have the following:

(i) Any strictly larger set $S_0' \supset S_0$ contains at least one zero-divisor a (let $ax = 0$, with $x \neq 0$). We then notice that $x/1 = (xa)/a = 0$ but $x \neq 0$, hence the natural homomorphism $A \longrightarrow S_0'^{-1}A$ cannot be injective.

(ii) An element of $S_0^{-1}A$ has the form a/s, where $a \in A$ and s is a non-zero divisor. We immediately see that if a is a zero divisor, then so is a/s and otherwise a/s is a unit. Hence, every element in $S_0^{-1}A$ is a zero divisor or a unit.

(iii) The natural homomorphism $f : A \longrightarrow S_0^{-1}A$ is injective, by (i). Now, given any element $a/s \in S_0^{-1}A$ the denominator s is not a zero divisor hence it must be a unit. If $ss' = 1$, then $a/s = as'/1 = f(as')$, hence f is also surjective. This completes the proof.

3.10

We have the following:

(i) If A is absolutely flat, then every principal ideal in A is idempotent, hence given $x \in A$ there is $\chi \in A$ such that $x(\chi x - 1) = 0$. Let a/s be any element of $S^{-1}A$ with $s(\sigma s - 1) = 0$ and $a(\alpha a - 1) = 0$. Then, we see that $(a/s)((\alpha/\sigma)(a/s) - 1) = 0$, hence $S^{-1}A$ is absolutely flat.

Actually, the two conditions are equivalent, if we assume the latter and take $S = \{1\}$.

(ii) Assume that A is absolutely flat. Then, given any maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is absolutely flat by (i). Given any non-zero, **non-invertible** $a/s \in A_{\mathfrak{m}}$ (namely $a \in \mathfrak{m}$, $s \in A - \mathfrak{m}$), there is $\alpha/\sigma \in A_{\mathfrak{m}}$ (namely $\alpha \in A$, $\sigma \in A - \mathfrak{m}$) such that $(a/s)((\alpha/\sigma)(a/s) - 1) = 0$. Obviously, $((\alpha/\sigma)(a/s) - 1) = (\alpha a - \sigma s)/(\sigma s)$ cannot be a unit. It should thus be an element of the maximal ideal of $A_{\mathfrak{m}}$ (which is a local ring), therefore we must have $\alpha a - \sigma s \in \mathfrak{m}$. But since $a \in \mathfrak{m}$, $\alpha a \in \mathfrak{m}$, we conclude that $\sigma s \in \mathfrak{m}$, which is absurd since $\mathfrak{m}$ is prime and none of s and σ belong to $\mathfrak{m}$.

Conversely, let $A_{\mathfrak{m}}$ be a field for all maximal ideals $\mathfrak{m}$ and let $x \in A$. Then, if x is a unit, (x) is obviously idempotent. Otherwise, there is a maximal ideal $\mathfrak{m}$ which contains x. Assume that $x(\alpha x - 1) \neq 0$ for all $\alpha \in A$. Then, the element $x(\alpha x - 1)/1$ is always invertible, hence there is $a/b \in A_{\mathfrak{m}}$ and $u \in A - \mathfrak{m}$ such that $ax(\alpha x - 1)u = bu$, which is a contradiction, since the left hand side belongs to $\mathfrak{m}$, while the right to $A - \mathfrak{m}$. Therefore, there is some $\alpha \in A$ such that $\alpha x^2 = x$, which means that (x) is idempotent, thus A is absolutely flat, as desired.

3.11

We have the following:

(i) $\Leftrightarrow$ (ii) The condition that $A/\mathfrak{R}$ is absolutely flat is equivalent to the statement that given any non-nilpotent $x \in A$, there is $a \in A$ such that $x(ax - 1)$ is nilpotent or, equivalently, such that $x(ax - 1)$ belongs to all prime ideals of A. Now, let $x \notin \mathfrak{p}$. Obviously x is not nilpotent but there is $a \in A$ such that $x(ax - 1)$ is; in particular, $\overline{x}(a\overline{x} - 1) = 0$ in $A/\mathfrak{p}$, but $\overline{x} \neq 0$. Since $A/\mathfrak{p}$ is an integral domain, the above implies $a\overline{x} = 1$, or that $\overline{x}$ is a unit. Since x was arbitrary, we obtain that $A/\mathfrak{p}$ is a field hence $\mathfrak{p}$ is maximal.

Conversely, if every prime ideal is maximal, then given $x \in A$ and $\mathfrak{p} \in \operatorname{Spec}(A)$, there is $a_{\mathfrak{p}}$ such that $x(a_{\mathfrak{p}}x - 1) \in \mathfrak{p}$. Therefore, we see that

$$x \prod_{\mathfrak{p} \in \operatorname{Spec}(A)} (a_{\mathfrak{p}}x - 1) \in \mathfrak{R}$$

and the product takes the form $x(bx - 1)$ for some $b \in A$. But then $\overline{x}(\overline{b}\overline{x} - 1) = \overline{0}$ in $A/\mathfrak{R}$. Since the class $\overline{x}$ was arbitrary, we conclude that $A/\mathfrak{R}$ is absolutely flat.

(ii) $\Leftrightarrow$ (iii) This equivalence follows from chapter 1, exercise 1.18, because $\overline{\{p\}} = V(p)$.

(iv) $\Leftrightarrow$ (ii) In this case, we see that $X_{\mathfrak{p}}$ and $X_{\mathfrak{q}}$ partition X and separate $\mathfrak{p}$ and $\mathfrak{q}$.

If the above conditions are fulfilled, then $\operatorname{Spec}(A)$ is compact (since the definition of quasi-compactness in the book seems to coincide with the usual definition of compactness and $\operatorname{Spec}(A)$ is always quasi-compact) and the proof of the last equivalence shows that $\operatorname{Spec}(A)$ is totally disconnected.

3.12

The condition that A is an integral domain guarantees that $T(M)$ is a submodule of M. We then have the following:

(i) This first statement is obvious.

(ii) If $x \in T(M)$ (say $xy = 0$, with $y \neq 0$ in A), then by the properties of module homomorphisms $yf(x) = f(yx) = 0$ and $y \neq 0$, hence $f(x) \in T(N)$, which of course means $f(T(M)) \subseteq T(N)$.

(iii) The first map in $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M''$ will definitely remain injective when passing on to $0 \longrightarrow M' \xrightarrow{\overline{f}} M \xrightarrow{\overline{g}} M''$; what remains to be shown is that $\operatorname{Im}\overline{f} = \ker \overline{g}$ remains true. Indeed, let $x \in \operatorname{Im}\overline{f} = \ker g$; this, together with $x \in T(M)$ (which follows from our previous assertion and (ii)), shows that

$x \in \ker \overline{g}$ and thus $\mathrm{Im}\overline{f} \subseteq \ker \overline{g}$. Conversely, let $y \in \ker \overline{g} \subseteq \ker g = \mathrm{Im}f$; this,together with the fact that f is injective (hence $0 = ay = af(x) = f(ax)$ if and only if $ax = 0$) shows that $y \in \mathrm{Im}\overline{f}$ and thus $\mathrm{Im}\overline{f} \supseteq \ker \overline{g}$. This completes the proof that $\mathrm{Im}\overline{f} = \ker \overline{g}$.

(iv) Note that if $S = A - \{0\}$, then the field of fractions K of A is merely $S^{-1}A$, hence there is an isomorphism between $K \otimes_A M$ and $S^{-1}M$. Therefore the map $\phi : M \longrightarrow K \otimes_A M$ that sends x to $1 \otimes x$ is equivalent to (in particular, it will have the same kernel as) the map $\psi : M \longrightarrow S^{-1}M$ that sends x to $x/1$. Note that $x/1 = 0$ if and only if there is a non-zero $s \in A$ such that $sx = 0$, thus if and only if $x \in T(M)$. This completes the proof.

3.13

First note the straightforward fact that $T(S^{-1}M) = S^{-1}(T(M))$. Then we have the following:

(i) $\Rightarrow$ (ii) $T(M_\mathfrak{p}) = T(M)_\mathfrak{p} = 0$

(ii) $\Rightarrow$ (iii) O.K.

(iii) $\Rightarrow$ (i) Assume that $T(M) \neq \varnothing$; say $x \in T(M)$ with $xy = 0$ and $x, y \neq 0$. Then, let $\mathfrak{m}$ be a maximal ideal that contains $\mathfrak{a} = \mathrm{Ann}(x)$ (therefore note that $\alpha x \neq 0$, $\alpha y \neq 0$ for all $\alpha \in A - \mathfrak{m}$). We easily see that $y(x/1) = 0$ in $M_\mathfrak{m}$, and $y \neq 0$, which contradicts our assumption that $M_\mathfrak{m}$ is torsion-free. Therefore M is torsion-free.

3.14

The condition of the book implies that $(M/\mathfrak{a}M)_\mathfrak{m} = 0$ for all the maximal ideals $\mathfrak{m}$ of $A/\mathfrak{a}$ (note of course that $\mathfrak{a} \subset \mathrm{Ann}(M/\mathfrak{a}M)$ hence we may regard $M/\mathfrak{a}M$ as an $A/\mathfrak{a}$-module). By proposition 3.8 this implies $m/\mathfrak{a}M = 0$, or $M = \mathfrak{a}M$.

3.15

We just repeat the hint of the book; it constitutes a full solution. Let $x_1, x_2, \ldots, x_n$ be a set of generators of $F = A^n$ and let $e_1, e_2, \ldots, e_n$ be a canonical basis. Define $\phi : F \longrightarrow F$ by letting $\phi(e_i) = x_i$; this homomorphism is obviously surjective. In order to prove injectivity, we may as well prove the statement for the $A_\mathfrak{m}$-module $F_\mathfrak{m}$ (where $\mathfrak{m}$ is any maximal ideal of A). Now $A_\mathfrak{m}$ is a local ring; let $k = A/\mathfrak{m}$ be its residue field. Also let N be the kernel of ϕ. Since F is a flat A-module (A is flat and the product of any number of flat modules is flat too), the exact sequence $0 \longrightarrow N \xrightarrow{incl} F \xrightarrow{\phi} F \longrightarrow 0$ yields an exact sequence $0 \longrightarrow k \otimes N \xrightarrow{1\otimes incl} k \otimes F \xrightarrow{1\otimes\phi} k \otimes F \longrightarrow 0$. But $k \otimes F = k^n$ is an n-dimensional vector space and $1 \otimes \phi$ is surjective thus injective, thus $k \otimes N = 0$. But both k and N are finitely generated (the latter by chapter 2, exercise 12), thus $N = 0$ (since we can't obviously have $k = 0$). This completes the proof that ϕ is an isomorphism.

We deduce that every set of generators of F has at least n elements.

3.16

We have the following:

(i) $\Rightarrow$ (ii) This part of the problem is clear by proposition 3.16.

(ii) $\Rightarrow$ (iii) Assume that $\mathfrak{m}^e = (1)$ for some maximal ideal $\mathfrak{m}$ of A. By the assumption, there is $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{q}^c = \mathfrak{m}$. But, taking extensions yields $(1) = \mathfrak{m}^e = \mathfrak{q}^{ce} \subseteq \mathfrak{q}$, a contradiction.

(iii) $\Rightarrow$ (iv) We note that since B is flat, we only need to show the statement for an arbitrary finitely generated submodule M' of M. Indeed, we know that M is the direct limit of its finitely generated submodules $\{M_\alpha\}_{\alpha \in A}$, and direct limits commute with tensor products. Therefore,

$$\varinjlim (B \otimes_A M_\alpha) = B \otimes_A M$$

and since B is flat, $B \otimes_A M = 0$ if and only if $B \otimes_A M_\alpha = 0$ for some $\alpha \in A$ (this holds if we omit the tensor by B and continues to hold when we tensor by B, because the natural injective map that embeds M_α into M remains injective when we tensor by B). Assume that such a module M_α existed; by exercise 19, (viii) we see that $B \otimes_A M' = 0$ if and only if $f^{*-1}(\mathrm{Supp}(M)) = \varnothing$. But, since the support of M is non-empty (because M is non-zero) and there is at least one maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^e \neq (1)$, we arrive at a contradiction. This completes the proof of the statement.

(iv) $\Rightarrow$ (v) Let M' be the kernel of $M \longrightarrow M$ sending x to $1 \otimes x$. Then, the sequence $0 \longrightarrow M' \xrightarrow{incl} M \longrightarrow M_B$ is exact and since B is flat, so will the sequence $0 \longrightarrow M'_B \xrightarrow{1\otimes incl} M_B \longrightarrow (M_B)_B$ be. But, by chapter 2, exercise 13 the mapping $(M_B)_B \longrightarrow M_B$ is injective, hence $M'_B = 0$, which by our assumption yields $M' = 0$ (this is also a lemma in Lang's, *Algebra*).

(v) $\Rightarrow$ (i) If we let $M = A/\mathfrak{a}$, then our assumption implies that the mapping $M \longrightarrow M_B = B/\mathfrak{a}^e$ is injective. Pulling $B/\mathfrak{a}^e$ back to A yields $A/\mathfrak{a} = A/\mathfrak{a}^{ec}$ whence $\mathfrak{a} = \mathfrak{a}^{ec}$.

B is said to be *faithfully flat* over A.

3.17

Given any injective map $f : M' \longrightarrow M$ (where M', M are A-modules) and any A-module N, we observe that

$$f \otimes 1 \otimes 1 \longrightarrow (M' \otimes_B N) \otimes C \longrightarrow (M \otimes_B N) \otimes_C C$$

will be injective by the conditions of the problem and the canonical isomorphisms $(M' \otimes_B N) \otimes C \simeq M' \otimes_C (M \otimes_B C), (M \otimes_B N) \otimes C \simeq M \otimes_C (M \otimes_B C)$. But then the flatness of C implies that $f \otimes_B : M' \otimes_B N \longrightarrow M \otimes_B N$ is injective, and since N was arbitrary, it implies that B is a flat A-algebra.

3.18

We just repeat the hint of the book; it constitutes a full solution. By the given assumptions, $B_\mathfrak{p}$ will be flat over $A_\mathfrak{p}$, because flatness is a local property, and so will $B_\mathfrak{q}$ over $B_\mathfrak{p}$ (because the former is a localized version of the latter and $S^{-1}A$ is always flat over A). Therefore, $B_\mathfrak{q}$ is flat over $A_\mathfrak{p}$. We now note that the maximal ideal of $A_\mathfrak{p}$ extends to the maximal ideal of $B_\mathfrak{q}$, hence in particular its extension doesn't coincide with (1). Thus, by exercise 16, we deduce that the mapping $\mathrm{Spec}(B_\mathfrak{q}) \longrightarrow \mathrm{Spec}(A_\mathfrak{p})$ is surjective.

3.19

We have the following:

(i) It's obvious that $M \neq 0$ if and only if $\mathrm{Supp}(M) \neq \varnothing$ by proposition 3.8.

(ii) Let $M = A$ in (vii).

(iii) If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence, then so is $0 \longrightarrow M'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow M''_\mathfrak{p} \longrightarrow 0$ and obviously $M_\mathfrak{p}$ is non-trivial if and only if either of $M'_\mathfrak{p}$ and $M''_\mathfrak{p}$ is non-trivial; this implies $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$, as desired.

(iv) We see that $M_\mathfrak{p} = (\sum M_i)_\mathfrak{p} = \sum (M_i)_\mathfrak{p}$ (to see this, just bring an arbitrary element of the right-hand side over a common denominator) and the left-hand side is non-trivial if and only if at least one of the $(M_i)_\mathfrak{p}$ is and this yields $\mathrm{Supp}(M) = \bigcup \mathrm{Supp}(M_i)$, as desired.

(v) Let $\mathfrak{a} = 0$ in (vii).

(vi) By chapter 2, exercise 3, $(M \otimes_A N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ is non-trivial if and only if both $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are non-trivial, hence $\mathrm{Supp}(M \otimes_A N) = \mathrm{Supp}(M) \cap \mathrm{Supp}(N)$, as desired.

(vii) We see that $\mathfrak{p} \in \mathrm{Supp}(M/\mathfrak{a}M) \iff (M/\mathfrak{a}M)_\mathfrak{p} \neq 0$. This is equivalent to the existence of a class $\overline{x} \in M/\mathfrak{a}M$ such that $\overline{x}k \neq \overline{0}$ for all $k \in A - \mathfrak{p}$, which in turn is equivalent to the existence of an element $x \in M$ such that $xk \notin \mathfrak{a}M$ for all $k \in A - \mathfrak{p}$. It is straightforward that if there exists such an x, then $\mathfrak{p} \supseteq \mathfrak{a} + \mathrm{Ann}(M)$. Conversely, if no element of $\mathfrak{a} + \mathrm{Ann}(M)$ is contained in $A - \mathfrak{p}$, then we see that $kx \notin \mathfrak{a}M$ (where x is any element of $M - \mathfrak{a}M$). This completes the proof that $\mathrm{Supp}(M/\mathfrak{a}M) = V(\mathfrak{a} + \mathrm{Ann}(M))$, as desired.

(viii) We see that $\mathfrak{p} \in f^{*-1}(\mathrm{Supp}(M))$ if and only if $M_{f^*\mathfrak{q}} \neq 0$ and this is equivalent to $B_\mathfrak{q} \neq 0$ and $M_\mathfrak{q} \neq 0$. Hence, $f^{*-1}(\mathrm{Supp}(M)) = \mathrm{Supp}(B) \cap \mathrm{Supp}(M) = \mathrm{Supp}(B \otimes_A M)$, as desired.

3.20

We have the following:

(i) It's obvious that every prime ideal of A is a contracted ideal if and only if $f^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is surjective.

(ii) Given any prime ideal $\mathfrak{q} = \mathfrak{a}^e$ of $\mathrm{Spec}(B)$, its image under f^* will be $\mathfrak{a}^{ec}$, thus $\mathfrak{q} = \mathfrak{a}^{ece} = f^*(\mathfrak{q})^e$. Therefore, if $f^*(\mathfrak{q}) = f^*(\mathfrak{q}')$, then clearly $\mathfrak{q} = \mathfrak{q}'$ and f^* is injective.

The converse is not true. For example, consider the situation of chapter 1, exercise 21: the mapping ϕ^* is injective (in fact, it's bijective), but the ideal $\mathfrak{q}_2 = (A/\mathfrak{p}) \times 0$ is not extended. Otherwise, there would be an ideal $\mathfrak{a}$ of A such that $\phi(\mathfrak{a}) \subseteq \mathfrak{q}_2$ and then for all $a \in \mathfrak{a}$, $a/1 = 0$ in k, which yields $a = 0$ since A is an integral domain.

3.21

We have the following:

(i) By chapter 1, exercise 21 (and since ϕ^* is surjective onto its image) we deduce that $\phi^* : \mathrm{Spec}(S^{-1}A) \longrightarrow V(\ker(\phi))$ is a homeomorphism.

In particular (and this is obvious as well), the image of $\mathrm{Spec}(A_f)$ is the basic open set X_f.

(ii) To see that $S^{-1}f^* : \mathrm{Spec}(S^{-1}B) \longrightarrow \mathrm{Spec}(S^{-1}A)$ is the restriction of f^* on $S^{-1}Y$ we just need to observe that $S^{-1}f^*$ by definition maps $f(S)^{-1}\mathfrak{q} \in \mathrm{Spec}(f(S)^{-1}B)$ to $S^{-1}f^*(f(S)^{-1})\mathfrak{q} = S^{-1}f^*(\mathfrak{q}) \in \mathrm{Spec}(S^{-1}A)$, hence the result after the restrictions described in the exercise. The fact that $S^{-1}Y = f^*(S^{-1}X)$ is a corollary of the previous observation and the surjectivity of f^*, established in (i).

(iii) This follows directly from questions (ii) and (iii) and the obvious fact that $\mathfrak{q} \supseteq \mathfrak{b}$ is and only if $\overline{f}^*(\mathfrak{q}) \supseteq \mathfrak{a}$.

(iv) By the previous questions we readily obtain

$$f^{*-1}(\mathfrak{p}) = \mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}).$$

What now remains to be shown is that

$$\mathrm{Spec}(B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}) = \mathrm{Spec}(k(\mathfrak{p}) \otimes_A B),$$

where $k(\mathfrak{p})$ is the residue field at $\mathfrak{p}$. We know from the hint of exercise 16 that $B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \simeq (A_\mathfrak{p}/\mathfrak{p}^c) \otimes_A B$, but since $A_\mathfrak{p}$ is a local ring, the only possible contraction $\mathfrak{p}^c$ of $\mathfrak{p}$ is $\mathfrak{m}$, the maximal ideal of $A_\mathfrak{p}$. This completes the proof, since $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{m}$.

3.22

We see that the canonical image of $\mathrm{Spec}(A_\mathfrak{p})$ in $\mathrm{Spec}(A)$ is the set of all ideals contained in $\mathfrak{p}$; but that's exactly the intersection $\bigcap_{f \notin \mathfrak{p}} X_f$ of all open neighborhoods of $\mathfrak{p}$ in $\mathrm{Spec}(A)$.

3.23

We have the following:

(i) If $U = X_f = X_g$, then $A_f \simeq A_g$ (by the obvious mapping $a/f^n \mapsto a/g^n$), and this in particular implies that $A(U)$ is well defined and depends only on U.

(ii) If $X_g = U' \subseteq U = X_f$, then we know that implies $r(f) = r(g)$, hence there is $n \in \mathbb{N}$ such that $g^n = uf$ for some $u \in A$. Choose the minimal such n. This choice induces a *restriction homomorphism* $\rho : A(U) \longrightarrow A(U')$ defined by $a/f^m \mapsto a/g^{mn}$ and the map is well0defined by the minimality of $n \in \mathbb{N}$ and depends only on U and U'.

(iii) It's obvious, by definition, that $\rho\mid_{UU} = \mathrm{id}_U$.

(iv) It's obvious, by definition, that $U \supseteq U' \supseteq U''$ implies $\rho_{UU''} = \rho_{U'U''} \circ \rho_{UU'}$, hence the given diagram is commutative.

(v) Consider the homomorphism

$$\phi : \bigoplus_{f \notin \mathfrak{p}_x} A_f \longrightarrow A_{\mathfrak{p}}$$

defined by

$$\phi((\frac{a_f}{f^{n_f}})_{f \notin \mathfrak{p}_x}, 0, 0, \dots) = \frac{(\prod a_f)}{(\prod f^{n_f})},$$

where we include only the non-zero terms in the product above.

We observe that the map is surjective (this is obvious). For any element of the domain that is sent to $0/1 \in A_{\mathfrak{p}}$ there is $s \notin \mathfrak{p}$ such that

$$s \prod_f a_f = 0 \in \mathfrak{p}$$

and this implies that for at least one $f \notin \mathfrak{p}$, $a_f \in \mathfrak{p}$. This implies that $\frac{a_f}{f^{n_f}}$ belongs to the ideal generated by elements of the form $\frac{a'}{f^n} - \frac{a'u^m}{(f')^{nm}}$ (notation as in the problem itself); just let f, f' be units (in which case $n = 1$) and let $a' = af/(f^m - 1)$. Conversely, it's obvious that any element of this form will belong to the kernel of the map $\ker \phi$. By the first isomorphism theorem,

$$\bigoplus_{f \notin \mathfrak{p}_x} A_f / \ker \phi \simeq A_{\mathfrak{p}}.$$

But the left hand side is merely the direct limit of the $A(U) = A_f$; this completes the proof and establishes that

$$\varinjlim_{U \ni \mathfrak{p}_x} A(U) = A_{\mathfrak{p}}.$$

3.24

This result follows by the property of quasi-compactness (every open cover has a finite subcover) that $\mathrm{Spec}(A)$ enjoys. If $\{U_i\}_{1 \le i \le n}$ is a finite subcover of $\{U_i\}_{i \in I}$, then we observe that there is s such that $s = s_1, s_2$ in $A(U_1 \cap U_2)$, and inductively, if $s = s_1, s_2, \dots, s_n$ in $A(U_1, U_2, \dots, U_n)$, then $s = s_{n+1}$ will coincide with $s_1, s_2, \dots, s_{n+1}$ in $A(U_1 \cap U_2 \cap \dots U_{n+1})$ and we can construct a global s by this inductive procedure.

3.25

We reproduce the hint of the book; it constitutes a full proof. Let $\mathfrak{p}$ be a prime ideal of A and let $k = k(\mathfrak{p})$ be the residue field at $\mathfrak{p}$. Then, by exercise 21, the fibre $h^{*-1}(\mathfrak{p})$ is merely the spectrum of $(B \otimes_A C) \otimes_A k \simeq (B \otimes_A k) \otimes_k (C \otimes_A k)$. Hence, $\mathfrak{p} \in h^*(T)$ if and only if $(B \otimes_A k) \otimes_k (C \otimes) Ak) \neq 0$ if and only if $B \otimes_A k \neq 0$ and $C \otimes_A k \neq 0$ which is equivalent to $\mathfrak{p} \in f^*(Y) \cap g^*(Z)$. This completes the proof that

$$h^*(T) = f^*(Y) \cap g^*(Z).$$

3.26

Again, we will merely repeat the hint of the book; it constitutes a full proof. Let $\mathfrak{p}$ be a prime ideal of A. Then, the fibre $f^*(\mathfrak{p})$ is, by exercise 21, the spectrum

$$B \otimes_A k(\mathfrak{p}) = \varinjlim B \otimes_A k(\mathfrak{p}) = \varinjlim (B_\alpha \otimes_A k(\mathfrak{p})),$$

since direct limits commute with tensor products. Therefore, we obtain that $\mathfrak{p} \notin f^*(\mathrm{Spec}(B))$ if and only if $f^*(\mathfrak{p}) = \varnothing$ if and only if $B_\alpha \otimes_A k(\mathfrak{p}) = 0$ for some $\alpha \in A$, which is equivalent to $f^*_\alpha(\mathfrak{p}) = \varnothing$. This shows that $\mathfrak{p} \in f^*(\mathrm{Spec}(B))$ if and only if $\mathfrak{p} \in \cap f^*(\mathrm{Spec}(B_\alpha))$.

3.27

We have the following:

(i) This follows directly from exercises 25 and 26.

(ii) We observe that $\mathfrak{p} \in f^*(\mathrm{Spec}(B))$ if and only if there is $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{p} = f^*(\mathfrak{q})$ which is equivalent to the existence of $\alpha \in A$ such that $\mathfrak{p} \in f_\alpha^*(\mathrm{Spec}(B_\alpha))$. This establishes the desired formula

$$f^*(\mathrm{Spec}(B)) = f_\alpha^*(\mathrm{Spec}(B_\alpha)).$$

(iii) This is implies that the sets of the form $f^*\mathrm{Spec}(B)$ where $A \xrightarrow{f} B$ is any A-algebra form the closed subsets of a topology called the *constructible* topology.

(iv) Since the sets X_g are the basis of the constructible topology (as happens with the Zariski one), we may repeat the proof of chapter 1, exercise 17, (vii).

3.28

We have the following:

(i) Any open set X_g is the image $f^*(\mathrm{Spec}(A_g))$ of the induced map of $A \longrightarrow A_g$ given by $g \longmapsto g/1$, therefore it's closed.

(ii) The space $X_{C'}$ is, by definition, totally disconnected, and thus Hausdorff.

(iii) The identity mapping is indeed a continuous bijection $X_C \longrightarrow X_{C'}$. Since X_C is also compact (by the usual definition), the inverse of inclusion will also be continuous, therefore X_C is homeomorphic to $X_{C'}$.

(iv) This follows immediately from the previous questions.

3.29

If $F = g^*(\mathrm{Spec}(C)) \subseteq \mathrm{Spec}(B)$ is any closed set in the constructible topology of $\mathrm{Spec}(B)$, then $f^*(F) = (g \circ f)^*(\mathrm{Spec}(A))$ hence it's closed in the constructible topology of $\mathrm{Spec}(A)$. This implies that f^* is a *closed* mapping.

3.30

If the constructible and the Zariski topology coincide on $\mathrm{Spec}(A)$, then $\mathrm{Spec}(A)$ is compact (by exercise 28, (iv)) hence by exercise 11, $A/\mathfrak{R}$ is absolutely flat.

Conversely, if $A/\mathfrak{R}$ is absolutely flat, then by exercise 11 every prime ideal is maximal. Therefore, given any open set X_g, and any point $\mathfrak{p}$ in it, there is $g_\mathfrak{p} \in A$ such that $gg_\mathfrak{p} = 1$ in $A/\mathfrak{p}$ (of course because $A/\mathfrak{p}$ is a field). With this notation, $g \in \mathfrak{p}$ implies $g_\mathfrak{p} \notin \mathfrak{p}$, therefore

$$X_g = V\Big(\prod_{\mathfrak{p} \in \mathrm{Spec}(A)} (g_\mathfrak{p})\Big),$$

which shows that all open sets in the topology are simultaneously open and closed. Since the constructible topology is the minimal topology that achieves this (by exercise 28), we obtain that the Zariski topology is finer than the constructible one. But the converse is also true, therefore the two topologies coincide. This completes the proof.

Chapter 4

Primary Decomposition

4.1

If an ideal $\mathfrak{a}$ has primary decomposition, then the minimal ideals that contain $\mathfrak{a}$ are the minimal ideals associated with it; in particular, their number is finite (less than the number of ideals associated with $\mathfrak{a}$). But the subspaces $V(\mathfrak{p})$, where $\mathfrak{p}$ is minimal, are exactly the irreducible components of $\operatorname{Spec}(A) = \operatorname{Spec}(A/\mathfrak{a})$ equipped with the Zariski topology.

4.2

If $\mathfrak{a} = r(\mathfrak{a})$, then $\mathfrak{a}$ is an intersection of prime ideals by chapter 1, exercise 9, say

$$\mathfrak{a} = \bigcap_{i \in I} \mathfrak{p}_\mathfrak{i}.$$

Without loss of generality, we may assume this decomposition to be a minimal one, and then in particular we'll have $\mathfrak{p}_\mathfrak{i} \not\subseteq \mathfrak{p}_\mathfrak{j}$ if $i \neq j$. Since the $\mathfrak{p}_\mathfrak{i}$ are prime, we'll have $r(\mathfrak{p}_\mathfrak{i}) = \mathfrak{p}_\mathfrak{i}$ for each one of them. By the minimality of the decomposition, we see that all the $\mathfrak{p}_\mathfrak{i}$ are necessarily minimal, and therefore $\mathfrak{a}$ has no embedded prime ideals.

4.3

Assume that A is absolutely flat and $\mathfrak{p}$ is any primary ideal of A. Let x be any element of $A - \mathfrak{p}$. We see that $\overline{x} \neq \overline{0}$ in $A/\mathfrak{p}$ and by the absolute flatness of A there is $a \in A$ such that $x(ax-1) = 0 \in \mathfrak{p}$; in particular, $\overline{x}(\overline{ax} - \overline{1}) = \overline{0}$. But then $\overline{ax} - 1$ is nilpotent in $A/\mathfrak{p}$ and therefore $\overline{ax}$ is a unit by chapter 1, exercise 1. This implies that $\overline{x}$ is unit and thus $A/\mathfrak{p}$ is a field and $\mathfrak{p}$ is maximal.

4.4

The elements of $\mathfrak{m} = (2, t)$ are of the form $2k + tf(t)$, where $f \in \mathbb{Z}[x], k \in \mathbb{Z}$; this ideal is obviously maximal (the quotient $\mathbb{Z}[t]/\mathfrak{m}$ is isomorphic to $\{+1\}$, which is a field). The ideal $\mathfrak{q} = (4, t)$ is not a power of $\mathfrak{m}$; the contrary assumption leads to a contradiction because $\mathfrak{q}$ contains 3, which no power of $\mathfrak{m}$ contains. However, $\mathfrak{q}$ is primary, as we easily see. Finally, $g(x)^n = 4k + tf(t)$, for some $n \in \mathbb{N}, k \in \mathbb{Z}$ (which is equivalent to $g \in \mathfrak{q}$), if and only if $g(x) = 2k + tf^*(t) \in \mathfrak{m}$, which shows that $\mathfrak{q}$ is $\mathfrak{m}$-primary.

4.5

The ideals $\mathfrak{p}_1, \mathfrak{p}_2$ are obviously prime (by exercise 8), while $\mathfrak{m}$ is maximal because $k[x_1, x_2, \ldots, x_n]/\mathfrak{m} \simeq k$, a field. An element T of $k[x, y, z]$ belongs to $\mathfrak{a} = \mathfrak{p}_1\mathfrak{p}_2$ if and only if:

$$T = x^2 f(x, y, z) + xyg(x, y, z) + xzh(x, y, z) + yzp(x, y, z),$$

for suitable polynomials f, g, h, p. This is equivalent to $T \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$, establishing the desired equality.

4.6

The answer turns out to be yes, 0 has primary decomposition in $C(X)$. For the proof, we will first need a lemma: any non-unit of $C(X)$ is a zero divisor (note, however, that no nonzero element of $C(X)$ is nilpotent!). Indeed, if $f \in C(X)$ is not a unit, then the set Z of zeroes of f in X is non-empty (otherwise, $g = \frac{1}{f}$ is the inverse of f); say $x_0 \in Z$. If Z has non-empty interior, then by Urysohn's lemma there is a map $g \in C(X)$ such that g is 0 outside the interior of Z and 1 at some fixed point in the interior of Z; in particular, $fg = 0$ everywhere on X, but $g \neq 0$.

If the interior of Z is empty, then $X - \overline{Z}$ is dense in X, hence f cannot be 0 everywhere in $X - \overline{Z}$ (a continuous function is 'determined' by its values at a set dense to its domain). Actually, Z must be closed (since f is continuous), hence we may replace $\overline{Z}$ by Z above. Let $x_0' \notin Z$ such that $f(x_0') \neq 0$. Now, since X is compact and Z is closed, Z will be compact too; take, for every $x \in Z$ an elementary neighborhood U of x; we keep the (finite) number of those $x^{(n)}$ whose neighborhoods cover Z. Now, assume that $\{U^{(n)}\}$ is such a cover of Z and order the set of open neighborhoods of each $x^{(n)}$ by inclusion and let $U_i^{(n)}, \supseteq$ be any infinite descending chain. Pick points $x_i^{(n)}$ in $U_i^{(n)}$, so that $x_i \neq x_j$ unless $i = j$ (we can do that since the space is assumed normal). For each point, define (by Urysohn's lemma) $f_i^{(n)} \in C(X)$ such that $f_i^{(n)}$ is 0 outside $U_i^{(n)}$ (in particular at x_0') and 1 at x_i. Note that the filter $\{x_i^{(n)}\}_{i \in I}$ converges to $x^{(n)}$ for all $n \in \mathbb{N}$ (by the very definition of compactness) and since the $f_i^{(n)}$ are continuous $f_i^{(n)}(x_i^{(n)}) \xrightarrow{i \in I} f^{(n)}(x^{(n)})$ (for fixed n). The product of all these maps except for one of them, say $f^{(1)}$, yields a map $g \in C(X)$ that has the desired property of $g \neq 0$ and $gf \equiv 0$ on X.

We now know that the set of all zero-divisors is the union of prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ (by the fact that its complement is saturated and chapter 3, exercise 7); note that these prime ideals are primary, of course. Since

$$C(X) = \bigcup_{i=1}^{n} \mathfrak{p}_i,$$

we deduce that

$$0 = \bigcap_{i=1}^{n} \mathfrak{p}_i,$$

and this presentation provides a primary decomposition of 0.

4.7

We have the following:

(i) By definition, $\mathfrak{a}^e = A[x]\mathfrak{a} = \mathfrak{a}[x]$, as desired.

(ii) This is implied by the following fact we proved in chapter 2, exercise 6: $(A/\mathfrak{a}) \simeq A[x]/\mathfrak{a}[x]$ for all ideals $\mathfrak{a}$ of A. This implies that if $A/\mathfrak{p}$ is an integral domain, then so is $A[x]/\mathfrak{p}[x]$, and therefore if $\mathfrak{p}$ is a prime ideal of A, then so is $\mathfrak{p}[x]$.

(iii) Given any ring B (in particular, $A[x]/\mathfrak{q}[x]$), the property P: "$B \neq 0$ and every zero divisor of B is nilpotent" is invariant under isomorphisms, therefore the identification of the previous exercise yields that $\mathfrak{q}[x]$ would be primary if $(A/\mathfrak{q})[x]$ satisfied P. But $\mathfrak{q}$ is primary in A, therefore $(A/\mathfrak{q})[x] \neq 0$ and given any zero-divisor $f \in (a/\mathfrak{q})[x]$, there is $a \in A/\mathfrak{q} - \{\overline{0}\}$, such that $af = 0$ (by chapter 1, exercise 2, (iii)). This implies, since $\mathfrak{q}$ is primary, that all the coefficients of f are nilpotent, hence that f itself is nilpotent, by chapter 1, exercise 2, (ii).

The only thing that is needed to complete the proof is the equality $r(\mathfrak{q}[x]) = r(\mathfrak{q})[x] = \mathfrak{p}[x]$, but this follows immediately from the isomorphism of the previous question and, in particular, the identification of their radicals (this follows again from chapter 1, exercise 2).

(iv) Certainly if

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

is a minimal primary decomposition of $\mathfrak{a} \subseteq A$, then $\mathfrak{a}[x] = \bigcap_{i=1}^{n} \mathfrak{p}[x]$ is a primary decomposition in $A[x]$, by the previous questions. This will also be minimal, because the radicals of the components obviously remain distinct and

$$\mathfrak{q}_i \not\supseteq \bigcap_{j=1, j\neq i}^{n} \mathfrak{q}_j$$

implies

$$\mathfrak{q}_i[x] \not\supseteq \bigcap_{j=1, j\neq i}^{n} \mathfrak{q}_j[x],$$

which completes the proof.

(v) This is obvious by the correspondence established in the previous question.

4.8

By the previous exercise, we may consider without loss of generality the case $\mathfrak{p} = (x)$ in $k[x]$. It's obvious that $\mathfrak{p}$ is prime and all its powers are primary (it's a simple matter of divisibility).

4.9

If $x \in A$ is a zero divisor, then $x \in \mathfrak{a} = (0 : a)$ for some non-zero $a \in A$ and the set of prime ideals containing $\mathfrak{a}$ is non-empty (since $a \neq 0$); therefore, there are indeed minimal prime ideals that contain $\mathfrak{a}$. If $\mathfrak{p}$ is one of those, then $x \in \mathfrak{p} \in D(A)$.

Conversely, if $x \in \mathfrak{p}$ and $\mathfrak{p}$ is minimal in the set of all prime ideals that contain $\mathfrak{a} = (0 : a)$ for some $a \in A$, then x is contained in all prime ideals that contain $\mathfrak{a}$, therefore, $x \in r(\mathfrak{a})$. This implies that there is a minimal $n \in \mathbb{N}$ such that $x^n \in \mathfrak{a}$, hence $x(x^{n-1}a) = 0$ and $x^{n-1}a \neq 0$, by the definition of n; this means that x is a zero-divisor.

We see that $\mathfrak{p} \in D(S^{-1}A)$ is equivalent to $\mathfrak{p} = S^{-1}\mathfrak{q}$ for some $\mathfrak{q} \in \operatorname{Spec}(A)$ that doesn't meet S. But then, since $S^{-1}(0 : a) = 0 : S^{-1}a$, and $\mathfrak{q}$ is a necessarily a prime ideal, we deduce the desired equality:

$$D(S^{-1}A) = D(A) \cap \operatorname{Spec}(S^{-1}A).$$

The fact that $D(A)$ is the set of ideals associated with 0 (in case, of course, 0 has primary decomposition) is obvious; it follows from the first Uniqueness Theorem and proposition 4.7.

4.10

From the definition of $S_{\mathfrak{p}}(0)$, it is clear that $S_{\mathfrak{p}}(0) = \{x \in A : \text{there is some } k \in A - \mathfrak{p} \text{ such that } xk = 0\}$. We thus have the following:

(i) If $x \in S_{\mathfrak{p}}(0)$, then $xk = 0 \in \mathfrak{p}$ and since $k \notin \mathfrak{p}$, we must have $x \in \mathfrak{p}$. Therefore, $S_{\mathfrak{p}}(0) \subseteq \mathfrak{p}$.

(ii) If $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$ and $\mathfrak{p}' \subset \mathfrak{p}$, then choose $x \in \mathfrak{p}$ such that $x \notin \mathfrak{p}'$. Then, there is $k \notin \mathfrak{p}$ (in particular, $k \notin \mathfrak{p}'$) and $n \in \mathbb{N}$ such that $x^n k = 0 \in \mathfrak{p}'$. But since neither x^n nor k are elements of $\mathfrak{p}'$, this is absurd. Therefore, $\mathfrak{p}$ is minimal.

Conversely, by exercise 11, $S_{\mathfrak{p}}(0)$ is a primary ideal, hence its radical will be prime. But, if $r(S_{\mathfrak{p}}(0)) \subseteq \mathfrak{p}$ and $\mathfrak{p}$ is minimal, we must have $r(S_{\mathfrak{p}}(0)) = \mathfrak{p}$.

(iii) This is obvious from the definition of $S_{\mathfrak{p}}(0)$.

(iv) This part of the exercise doesn't hold unless 0 has primary decomposition (or, at least, holds vacuously). In case 0 does have primary decomposition, then by exercise 9 $D(A)$ is the set of prime ideals that belong to 0 hence the equality

$$0 = \bigcap_{\mathfrak{p} \in D(A)} S_{\mathfrak{p}}(0).$$

4.11

First of all we claim that $S_{\mathfrak{p}}(0)$ is primary. Indeed, let $xy \in S_{\mathfrak{p}}(0)$, which means there is $k \notin \mathfrak{p}$ such that $k(xy) = 0$. If $y \notin S_{\mathfrak{p}}(0)$ (in particular, $ky \neq 0$), then $x/1$ is a zero divisor in $A_{\mathfrak{p}}$. But since $\mathfrak{p}$ is a minimal prime ideal, there are no prime ideals in $A_{\mathfrak{p}}$ other than the unique maximal ideal $\mathfrak{m}$, hence $\mathfrak{R}_{A_{\mathfrak{p}}} = \mathfrak{m}$ and the set of zero-divisors (generally a union of prime ideals) also coincides with $\mathfrak{m}$. We deduce that $x/1$ is in fact nilpotent, and this means that $x^n \in S_{\mathfrak{p}}(0)$ for some $n \in \mathbb{N}$. Therefore, $S_{\mathfrak{p}}(0)$ is primary.

Now we will show that $S_{\mathfrak{p}}(0)$ is contained in all $\mathfrak{p}$-primary ideals. Indeed, if $\mathfrak{q}$ is $\mathfrak{p}$-primary, then $\mathfrak{q} \subseteq \mathfrak{p}$, hence if $x \in S_{\mathfrak{p}}(0)$, with $kx = 0 \in \mathfrak{q}$ (where $k \notin \mathfrak{p}$, hence in particular $k \notin \mathfrak{p}$) then if $x \notin S_{\mathfrak{p}}(0)$ we would have $k^n \in S_{\mathfrak{p}}(0)$ (for some $n \in \mathbb{N}$) hence $k^n \in \mathfrak{p}$, which is absurd. Therefore, $x \in \mathfrak{q}$ and thus $S_{\mathfrak{p}}(0) \subseteq \mathfrak{q}$, as desired.

Exercise 10, (ii) now guarantees that $S_{\mathfrak{p}}(0)$ is $\mathfrak{p}$-primary. Note that the above conditions imply that

$$S_{\mathfrak{p}}(0) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \mathfrak{q}_n,$$

where $\mathfrak{q}_i, 1 \leq i \leq n$ are the $\mathfrak{p}$-primary ideals of A.

This fact implies that $\mathfrak{a}$, as defined in the exercise, is contained in all the prime ideals of A (since it's contained in all $\mathfrak{p}$-primary ideals, in particular in $\mathfrak{p}$), therefore it's also contained in their intersection, the nilradical of A.

The last part of the problem follows easily:

$$0 = \bigcap_{r(\mathfrak{q}_\mathfrak{i} = \mathfrak{p}_\mathfrak{i})} \mathfrak{q}_\mathfrak{i},$$

where the intersection runs over all the minimal ideals of A (and note that this is a minimal primary decomposition), if and only if all the prime ideals associated with 0 are minimal, which means that 0 contains only isolated ideals.

4.12

By definition, $S(\mathfrak{a}) = (S^{-1}\mathfrak{a})^c$. We thus have the following (we use freely the results of chapter 1, proposition 1.18):

(i) $S(\mathfrak{a}) \cap S(\mathfrak{b}) = (S^{-1}\mathfrak{a})^c \cap (S^{-1}\mathfrak{b})^c = (S^{-1}(\mathfrak{a} \cap \mathfrak{b}))^c = (S^{-1}\mathfrak{a} \cap S^{-1}\mathfrak{b})^c = S(\mathfrak{a} \cap \mathfrak{b})$, the last equality following from the commutativity of the S^{-1} functor with most reasonable operators.

(ii) This second result follows similarly since all the operators commute.

(iii) We see that $S(\mathfrak{a})^c = (1)$ if and only if $S^{-1}\mathfrak{a} = (1)$, and that's equivalent to $\mathfrak{a} \cap S \neq \varnothing$.

(iv) The last property follows from the canonical identification $\phi : S_1^{-1}(S_2^{-1}A) \longrightarrow (S_1 S_2)^{-1}A$ given by $(a/s_1)/s_2 \mapsto a/s_1 s_2$. This is, in fact, an isomorphism and yield the desired result.

If $\mathfrak{a}$ has a primary decomposition and S is any multiplicatively closed subset of A, then

$$S(\mathfrak{a}) = \bigcap_{i=m+1}^{n} \mathfrak{q}_\mathfrak{i},$$

where $\mathfrak{q}_\mathfrak{i}$, $1 \leq i \leq n$, are those primes associates of $\mathfrak{a}$ whose radicals don't meet S (by proposition 4.9). Therefore, the number of discrete sets if the form $S(\mathfrak{a})$ is at most 2^n; in particular, it's finite.

4.13

(i) Notice that by exercise 4.11, the ideal $S_{\mathfrak{p}/\mathfrak{p}^n}(0)$ is primary in $A/\mathfrak{p}^n$, because $r(\mathfrak{p}^n) = \mathfrak{p}$, hence $\mathfrak{p}$ is a minimal prime over $\mathfrak{p}^n$. Contracting $S_{\mathfrak{p}/\mathfrak{p}^n}(0)$ to A, yields that A is $\mathfrak{p}$-primary.

(ii) Since $\mathfrak{p}$ is minimal over $\mathfrak{p}$, this follows from corollary 4.11.

(iii) Since $\mathfrak{p}$ is minimal over $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$, corollary 4.11 yields again that $S_{\mathfrak{p}}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})$ is the $\mathfrak{p}$-primary component of $\mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$. It remains to be shown that $\mathfrak{p}^{(m+n)} = S_{\mathfrak{p}}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})$. We easily observe that $\mathfrak{p}^{m+n} = \mathfrak{p}^m\mathfrak{p}^n \subseteq \mathfrak{p}^{(m)}\mathfrak{p}^{(n)}$ implies $\mathfrak{p}^{(m+n)} \subseteq S_{\mathfrak{p}}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})$.

Conversely, let $x \in S_{\mathfrak{p}}(\mathfrak{p}^{(m)}\mathfrak{p}^{(n)})$. Then, there exists $s \in \mathfrak{p}$ such that $sx = \sum_i' x_i y_i$, with $x_i \in \mathfrak{p}^{(m)}, y_i \in \mathfrak{p}^{(n)}$. So, in turn for each i there are $s_i, t_i \in S_{\mathfrak{p}}$ such that $s_i x_i \mathfrak{p}^m, t_i y_i \in \mathfrak{p}^n$. Then,

$$s(\prod_j s_j)(\prod_k t_k)x = (\prod_j s_j)(\prod_k t_k)\sum_i x_i y_i \in \mathfrak{p}^m\mathfrak{p}^n = \mathfrak{p}^{m+n},$$

therefore $x \in S_{\mathfrak{p}}(\mathfrak{p}^{m+n}) = \mathfrak{p}^{(m+n)}S$. This completes the proof.

(iv) If $\mathfrak{p}^{(n)} = \mathfrak{p}^n$, then $\mathfrak{p}^n$ is primary, by question (i). Conversely, if $\mathfrak{p}^n$ is primary, then $\mathfrak{p}^n = \mathfrak{p}^n$ is a primary decomposition of $\mathfrak{p}^n$, and its $\mathfrak{p}$-primary component $\mathfrak{p}^n$ is equal to $\mathfrak{p}^{(n)}$, by question (ii).

4.14

If $\mathfrak{p}$ is prime, then it's obviously going to be a prime ideal associated with $\mathfrak{a}$. Without loss of generality, assume $\mathfrak{a} = 0$, by passing on to $A/\mathfrak{a}$ (note that if the image $\overline{\mathfrak{p}}$ of $\mathfrak{p}$ through the natural projection $\pi : A \longrightarrow A/\mathfrak{a}$ is prime in $A/\mathfrak{a}$, then its contraction $\mathfrak{p}$ will be prime in A). Now, assume that $\mathfrak{p} = (0 : x) = \mathrm{Ann}(x)$. Given $a \notin \mathrm{Ann}(x)$, we notice that $\mathrm{Ann}(x) = \mathrm{Ann}(ax)$, since $\mathrm{Ann}(x) \subseteq \mathrm{Ann}(ax)$ and $\mathrm{Ann}(x)$ is maximal among annihilators of elements of A. But then, if $yz \in \mathrm{Ann}(x)$, and $y \notin \mathrm{Ann}(x)$, we would have $\mathrm{Ann}(xy) = \mathrm{Ann}(x)$. The equation $0 = (yz)x = z(xy)$ implies $z \in \mathrm{Ann}(xy) = \mathrm{Ann}(x)$. Hence, $\mathfrak{p} = \mathrm{Ann}(x)$ is prime and this completes the proof.

4.15

Assume that the primary decomposition of $\mathfrak{a}$ is

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cdots \cap \mathfrak{q}_n,$$

where $r(\mathfrak{q}_i) = \mathfrak{p}_i$ and $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ constitute the isolated part of $\mathfrak{a}$ (these are minimal). It's easy to observe that the set of all such f's (with the notation of the problem) is exactly the set $(\mathfrak{q}_m \cap \mathfrak{q}_{m+1} \cdots \cap \mathfrak{q}_n) - (\mathfrak{q}_1 \cap \mathfrak{q}_2 \cdots \cap \mathfrak{q}_{m-1})$ and thus $\mathfrak{q}_\Sigma = S_f(\mathfrak{a})$ by proposition 4.9. The second equality $S_f(\mathfrak{a}) = (\mathfrak{a} : f^n)$ follows directly from the selection of f.

4.16

By proposition 4.9, any ideal of the form $S^{-1}\mathfrak{a}$ has primary decomposition

$$S^{-1}\mathfrak{a} = \bigcap_{i=1}^{m} S^{-1}\mathfrak{q}_i$$

if

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

and $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_m$ $(r(\mathfrak{q}_i) = \mathfrak{p}_i)$ are the primary ideals belonging to $\mathfrak{a}$ that don't meet S. But all ideals of $S^{-1}A$ are of this form for some $\mathfrak{a}$ that doesn't meet S, so in particular all ideals of $S^{-1}A$ will have primary decomposition if all ideals of A do.

4.17

We will essentially repeat the hint of the book; it constitutes a full solution. Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{p}_1$ be a minimal prime ideal that contains $\mathfrak{a}$. Then, by exercise 11, $\mathfrak{q}_1 = S_{\mathfrak{p}_1}(\mathfrak{a})$ is $\mathfrak{p}_1$-primary and $\mathfrak{q}_1 = (\mathfrak{a} : x)$ for some $x \in A - \mathfrak{p}_1$. Now, certainly $\mathfrak{a} \subseteq \mathfrak{q}_1 \cap (\mathfrak{a} + (x))$; conversely, if $y = a + tx \in \mathfrak{a} + (x)$ also belongs to $\mathfrak{q}_1$, then reducing modulo $\mathfrak{p}_1$, we obtain $tx = 0$ in $A/\mathfrak{p}_1$, hence (given $x \notin \mathfrak{p}_1$) $t \in \mathfrak{p}_1 = r(\mathfrak{q}_1) = (\mathfrak{a} : x)$, hence $tx \in \mathfrak{a}$, which means $a + tx \in \mathfrak{a}$. Therefore, $\mathfrak{a} = \mathfrak{q}_1 \cap ((\mathfrak{a}) + (x))$.

Now let $\mathfrak{a}_1$ be a maximal ideal of the set of ideals $\mathfrak{b}$ that contain $\mathfrak{a}$ and satisfy $\mathfrak{a}_1 \cap \mathfrak{b} = \mathfrak{a}$ and choose $\mathfrak{a}_1$ so that $x \in \mathfrak{a}_1$ hence $\mathfrak{a}_1 \nsubseteq \mathfrak{p}_1$. Repeating the construction with $\mathfrak{a}_1$ and so on yields (at the nth step) $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \mathfrak{q}_n \cap \mathfrak{a}_n$, where the $\mathfrak{q}_i$ are primary ideals and $\mathfrak{a}_n$ is maximal among the ideals $\mathfrak{b}$ that contain $\mathfrak{a}_{n-1} = \mathfrak{a}_n \cap \mathfrak{q}_n$. This implies that $\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n \cap \mathfrak{b}_n$, and $\mathfrak{a}_n \nsubseteq \mathfrak{p}_n$. If at any stage we have $\mathfrak{a}_n = (1)$, then the process stops and $\mathfrak{a}$ is the finite intersection of primary ideals. Otherwise, the process continues to yield new ideals and, as $n \longrightarrow \infty$, $\mathfrak{a}_n$ becomes 'arbitrarily small', in the sense that $\mathfrak{a}_{n-1} \subseteq \mathfrak{a}_n$.

4.18

We have the following:

(i) $\Rightarrow$ (ii) If every ideal $\mathfrak{a}$ has primary decomposition, then the number of ideals of the form $S(\mathfrak{a})$, where S is a multiplicatively closed subset of A, is finite (by exercise 12), therefore A satisfies property (L2). A also satisfies (L1); for the proof of this, note that $S \subseteq S'$ (where S, S' are multiplicatively closed subsets of A) implies $S'(\mathfrak{a}) \subseteq S(\mathfrak{a})$, hence if $S_1 \supseteq S_2 \supseteq \cdots \supseteq S_n$ implies $S_1(\mathfrak{a}) \supseteq S_2(\mathfrak{a}) \supseteq \cdots \supseteq S_n(\mathfrak{a})$ and any such decreasing sequence has $S_f(\mathfrak{a})$ as a lower bound (we adopt the notation of exercise 15 here; we can always pick a suitable f, according to the solution of that exercise). Therefore, by Zorn's lemma, it must terminate and this completes the proof.

(ii) $\Rightarrow$ (i) Conversely, if A satisfies (L1) and (L2), then $\mathfrak{a}$ is the intersection of a possibly infinite number of primary ideals by exercise 17. With the notation of the proof of exercise 17, $S_n = S_{\mathfrak{p}_1} \cap \cdots \cap S_{\mathfrak{p}_n}$ necessarily meets $\mathfrak{a}_n$ (since $\mathfrak{a}_n \nsubseteq \mathfrak{p}_n = A - S_{\mathfrak{p}_n}$), therefore $S_n(\mathfrak{a}_n) = (1)$, hence $S_n(\mathfrak{a}) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$. This implies that $\{S_n\}_{n \in \mathbb{N}}$ is a decreasing sequence of multiplicative subsets of A. By property (L2), this sequence must terminate and then $\mathfrak{a}_n = (1)$, which implies that $\mathfrak{a}$ has a primary decomposition, as desired.

4.19

The first part of the problem follows from the second paragraph of problem 11. For the second part, we will merely repeat the hint of the book; it constitutes a full solution and an elegant one at that. We will proceed by induction; for $n = 1$, the statement is trivial, because $\mathfrak{p}_1$ is obviously a $\mathfrak{p}_1$-primary ideal. Assume that the result holds for $n - 1$ and let without loss of generality $\mathfrak{p}_n$ be maximal in the set $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots \mathfrak{p}_n\}$. By our inductive hypothesis, there is an ideal $\mathfrak{b}$ and a minimal decomposition $\mathfrak{b} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_{n-1}$, such that each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary. If $\mathfrak{b} \subseteq S)\mathfrak{p}_n(0)$ and $\mathfrak{p}$ were any minimal ideal contained in $\mathfrak{p}_n$, then $S_{\mathfrak{p}_n} \subseteq S_{\mathfrak{p}}(0)$, which implies $\mathfrak{b} \subseteq S_{\mathfrak{p}}(0)$. Taking the radicals of both sides yields $\mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_{n-1} \subseteq \mathfrak{p}$, hence $\mathfrak{p}_i \subseteq \mathfrak{p}$, for some $1 \leq i \leq n$ by chapter 1, hence $\mathfrak{p}_i = \mathfrak{p}$ by the minimality of $\mathfrak{p}$. But this is a contradiction, since $\mathfrak{p}$ may be minimal, but no $\mathfrak{p}_i$ is supposed to be. This implies that $\mathfrak{b} \nsubseteq S_{\mathfrak{p}_n}(0)$ and therefore there is a $\mathfrak{p}_n$-primary ideal $\mathfrak{q}_n$ such that $\mathfrak{b} \nsubseteq \mathfrak{q}_n$. This implies that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{q}_n$ has associated prime ideals exactly the $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$, and this decomposition is minimal; this completes the inductive step and the proof.

Primary decomposition of Modules

4.20

We observe that $x \in r_M(N)$ if and only if $x^q M \subseteq N$ for some positive integer q; this is equivalent to $x \in (N : M)$ and $x^q \in \mathrm{Ann}(M/N)$, which yields that

$$r_M(N) = r(N : M) = r(\mathrm{Ann}(M/N)).$$

In particular, $r_M(N)$ is an ideal.

Now the formulas analogous to (1.13) can be easily generalized and proved in the case of r_M.

4.21

It's easy to show that if the module Q is primary in M, then $(Q : M)$ is a primary ideal of A, hence $r_M(Q)$ is a prime ideal $\mathfrak{p}$ (the proof is exactly the same as in the case of ideals). Similarly, we can show the analog of lemma (4.3): if the modules $\mathfrak{Q}_i$, $1 \le i \le n$, are $\mathfrak{p}$-primary in M, then so is their intersection as well as the analog of lemma (4.4): Let Q be a $\mathfrak{p}$-primary module in M and x an element of M. Under these conditions:

(i) if $x \in Q$, then $(Q : x) = M$

(ii) if $x \notin Q$, then $(Q : x)$ is $\mathfrak{p}$-primary, and therefore $r(Q : x) = \mathfrak{p}$

4.22

Again, the proof that in a minimal decomposition

$$N = \bigcap_{i=1}^{n} Q_i$$

of modules the ideals $\mathfrak{p}_i = r_M(Q_i)$ depend only on N, follows closely the proof for the special case of ideals.

4.23

As before, propositions (4.6)-(4.11) inclusive can be shown using exactly the same arguments that are used for the special case of ideals; these proofs are given in the book.

Chapter 5

Integral Dependence and Valuations

5.1

Any prime ideal $\mathfrak{q}$ of $f(A)$ will be of the form $f(p)$, where $\mathfrak{p} = \mathfrak{q}^c$ is prime in A. The integrality of B over $f(A)$ implies, by theorem 5.10, that any prime ideal of $f(A)$ can be decomposed as $\mathfrak{q} \cap f(A)$; the converse, namely that any ideal of that form is prime, is also trivially true. Therefore, if $V(\mathfrak{b})$ is any closed subspace of $\mathrm{Spec}(B)$, then $f^*(\mathfrak{b}) = V(\mathfrak{b}^c)$ which is closed in $\mathrm{Spec}(A)$. This shows that the induced mapping $f^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is a *closed* mapping.

5.2

We can factor $f : A \longrightarrow \Omega$ through the natural projection map $\pi : A \longrightarrow A/\mathfrak{p}$, where $\mathfrak{p} = \ker f$ is prime (it's the contraction of the prime ideal $(\mathbf{0})$ of Ω) and then through the inclusion of the integral domain $A/\mathfrak{p}$ into its field of fractions $K(A/\mathfrak{p})$ and the inclusion of that into Ω (obviously $K(A/\mathfrak{p})$ will be a subfield of Ω). On the other hand, by the Lying-Over Lemma, there is a prime ideal $\mathfrak{q}$ of B, whose contraction in A is $\mathfrak{p}$ and moreover $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. Embedding $B/\mathfrak{q}$ to its field of fractions $K(B/\mathfrak{q})$, we obtain that L will be *algebraic* over K; this is so because $B/\mathfrak{q}$ is integral over $A/\mathfrak{p}$. Now it suffices to extend the map $K(A/\mathfrak{p}) \longrightarrow \Omega$ to $K(B/\mathfrak{p}) \longrightarrow \Omega$.

This is done in the standard fashion, as follows: if $k \in K(B/\mathfrak{q})$, let $p \in K(A/\mathfrak{p})[x]$ be the minimal polynomial that vanishes at k; this induces a polynomial $h(x) \in \Omega[x]$ by pushing the coefficients into Ω via f. This polynomial has a root, denote it by $\overline{f}(k)$ (its the image of k under the extended map $\overline{f} : K(B/\mathfrak{q}) \longrightarrow \Omega$). Note in particular that $\overline{f}$ remains injective (it's the extension of the final inclusion map).

We repeat the process and thus produce a collection $\{(E, \overline{f}) : K(A/\mathfrak{p}) \leq E \leq K(B/\mathfrak{q}), \overline{f}\mid_K A/\mathfrak{p} = f\}$ to which we can extract a maximal element by a Zorn's lemma-type argument; call its domain E_0. We claim that $E_0 = K(B/\mathfrak{q})$. Indeed, $K(B/\mathfrak{q})$ is algebraic over E_0, so if the inclusion were strict, then we would be able to extend $\overline{f}_{E_0}$ to $E_0(k)$, where $k \in K(B/\mathfrak{q}) - E_0$, a contradiction. This completes the proof.

5.3

Since the integral elements of a ring over a subring constitute a ring (corollary 5.3) and the generators of $B' \otimes_A C$ are the elements of the form $b' \otimes_A c$, we merely need to show that $b' \otimes_A c$ is integral over $(f \otimes_A 1)(B' \otimes_A C)$. Indeed, if $b' \in B'$ satisfies the following integral equation over $f(B)$:

$$b'^n + f(b_{n-1})b'^{n-1} + \cdots + f(b_1)b' + f(b_0) = 0,$$

then we observe that $b' \otimes_A c$ satisfies the equation

$$(b' \otimes_A c)^n + (f(b_{n-1}) \otimes c)(b' \otimes c)^{n-1} + \cdots + (f(b_1) \otimes c^{n-1})(b' \otimes_A c) + f(b_0) \otimes_A c^n = 0$$

over $(f \otimes_A 1)(B' \otimes_A C)$; this completes the proof.

5.4

We merely repeat the counterexample constructed in the hint; consider the subring $k[x^2-1]$ of $k[x]$, where k is a field, and let $\mathfrak{n}=(x-1)$. Then, the restriction $\mathfrak{m}$ of $\mathfrak{n}$ to A is (x^2-1). Were $B_{\mathfrak{n}}$ integral over $A_{\mathfrak{m}}$, the element $\frac{1}{x+1}$ would satisfy an equation of the form:

$$\frac{1}{(x+1)^n}+\sum_{m=0}^{n-1}\frac{g_m(x^2-1)}{k_m(x^2-1)(x+1)^m}=0,$$

where the k_m are polynomials in x^2-1 that don't vanish at ± 1 . Multiplying both sides of the equation by $(x+1)^{n-1}$, and then letting $x=-1$ yields a contradiction.

5.5

We have the following:

(i) Assume that $u\in B$ is the inverse of $x\in A$ and let $u^n+a_{n-1}u^{n-1}+\cdots+a_0=0$ be the equation over A that u satisfies. Multiplying both sides by x^{n-1} yields $u=-(a_{n-1}+a_{n-2}x+\cdots+a_0X^{n-1})\in A$, as desired.

(ii) If $\mathfrak{m}$ is a maximal ideal of A, then Theorem 5.10 implies that there is a maximal ideal $\mathfrak{e}$ of B such that $\mathfrak{m}=\mathfrak{e}\cap A$; the converse, namely that any ideal of such form will be maximal, is obviously true. This clearly yields that $\mathfrak{J}_A=\mathfrak{B}\cap A$, as desired.

5.6

If $B_1,B_2,\ldots,B_n$ are all integral A-algebras, with corresponding mappings $f_I\longrightarrow A$, then $f_i(A)\overset{int.}{\subseteq}B$ and therefore given any $x_i\in B_i$, $f_i(A)[x]$ is a finitely generated B_i-module. This implies that, $(f_1(A)[x_1],f_2(A)[x_2],\ldots,f_2(A)[x$ is a finitely generated A-module (the number of its generators will equal at most the sum of the numbers of generators of the $f_i(A)$, which is finite). Therefore, $\prod B_i$ is an integral A-algebra, as desired.

5.7

Assume that $B-A$ is multiplicatively closed, but A is not integrally closed in B; let C then be its integral closure in B. Given any element $y\in C-A$, let n be the minimal degree of all polynomials in $A[x]$ that y satisfies; note that $n\geq 2$, since $y\notin A$. Then, if $y^n+a_{n-1}y^{n-1}+\cdots+a_0=0$, where $a_i\in A$ for $0\leq i\leq n$, we observe that

$$y^{n-1}+a_{n-1}y^{n-2}+\cdots+a_1\notin A,$$

by the minimality condition on n and since $y\notin A$, we obtain

$$y(y^{n-1}+a_{n-1}y^{n-2}+\cdots+a_1)\notin A,$$

by the multiplicative closure of $B-A$. However,

$$y(y^{n-1}+a_{n-1}y^{n-2}+\cdots+a_1)=-a_0\in A,$$

which is absurd. Therefore, A is integrally closed in B.

5.8

We have the following:

(i) We will merely repeat the hint of the book; it constitutes a full proof. Let $L\supseteq B$ be a field in which f,g split into linear factors; say $f=\prod(x-\xi_i),g=\prod(x-\eta_j)$. Then, the roots ξ_i,η_j are integral over C, hence so are the coefficients of f,g (because the integral closure of A is a subring of B). Since the coefficients also belong to B, they must necessarily belong to C (because it's integrally closed in B) and thus $f \in C[x], g \in B[x]$.

(ii) In this case, the only thing that needs to be shown is that there exists, in fact, a field extension of the general ring B, in which f, g split into linear factors. We will just sketch the construction, which is identical to the one for fields. For that, we will perform induction on the degree of f. If $\deg(f) = 1$, the claim is true. Assume it holds for some degree n and consider the ring $D = B[t]/(f)$, where (f) is the principal ideal generated by f in $B[t]$. Then, we can embed $B[t]$ naturally into D, by mapping any $h(t) \in B[t]$ to its class in D. In particular, the image $\bar{t}$ of $t \in B[t]$ will be a root of f in D, hence $f(x) = (x - \bar{t})f_1(x)$. The polynomial $f_1(x)$ will have degree $n-1$ and by our inductive hypothesis it will split into linear factors in some extension E of D; by the relation $f(x) = (x - \bar{t})f_1(x)$, so will f. This completes the proof.

5.9

Again, we will merely repeat the hint of the book, which constitutes a full proof. Let f be integral over $A[x]$; it will satisfy a relation of the form:

$$f^m + g_1 f^{m-1} + \cdots + g_m = 0,$$

where the g_i are elements of $A[x]$, of course. Then, let r be a positive integer greater than the degrees of all the g_i and m; put $f_1 = f - x^r$, so that

$$(f_1 + x^r)^m + g_1(f_1 + x^r)^{m-1} + \cdots + g_m = 0,$$

or rather

$$f_1^m + h_1 f_1^{m-1} + \cdots + h_m = 0;$$

here, $h_m = (x^r)^m + g_1(x^r)^{m-1} + \cdots + g_m \in A[x]$. This implies that $f_1(f_1^{m-1} + h_1 f_1^{m-2} + \cdots + h_{m-1}) \in A[x]$ and thus exercise 8 implies that $f_1 \in C[x]$. Since f was an arbitrary integral element of B, this deduction shows that $C[x]$ is the integral closure of $A[x]$ in $B[x]$, as desired.

5.10

Under the conditions of the problem, we have the following:

i)

(a) $\Rightarrow$ (c) If $f^* : \mathrm{Spec}(A/\mathfrak{p}) \longrightarrow \mathrm{Spec}(B/\mathfrak{q})$ is a closed mapping, then given $\mathfrak{p} = \mathfrak{q}^c$ as in (c), the closed set $\mathrm{Spec}(A/\mathfrak{p}) = V(\mathfrak{p})$ will be identified with the closed set $V(\mathfrak{p}^e) = V(\mathfrak{q}) = \mathrm{Spec}(B/\mathfrak{q})$.

(b) $\Rightarrow$ (c) If $\mathfrak{p} \subseteq \mathfrak{n}$ inside $\mathrm{Spec}(A/\mathfrak{p})$, then, the going-up property is equivalent to the existence of $\mathfrak{m}$ that contains $\mathfrak{q} = \mathfrak{p}^c$, such that $\mathfrak{m}^c = \mathfrak{n}$, or equivalently $f^*(\mathfrak{m}) = \mathfrak{n}$, or equivalently $f^* : \mathrm{Spec}(A/\mathfrak{p}) \longrightarrow \mathrm{Spec}(B/\mathfrak{q})$, or equivalently to the surjectivity of f^*.

(c) $\Rightarrow$ (b) Since we can obviously reduce the proof of the going-up property to the case of a chain of ideals of length 2 (as in its proof in the book), the argument above shows this direction too.

ii)

(a') $\Rightarrow$ (c') This is the dual of the '(a) $\Rightarrow$ (b)' statement above; we can thus prove it in exactly the same fashion, reversing the arrows and substituting $\mathrm{Spec}(B/\mathfrak{q}), \mathrm{Spec}(A/\mathfrak{p})$ with $\mathrm{Spec}(B_\mathfrak{q}), \mathrm{Spec}(A_\mathfrak{p})$ respectively.

(b') $\Rightarrow$ (c') This is the dual of the '(b) $\Rightarrow$ (c)' statement above; we can thus prove it in exactly the same fashion, reversing the arrows.

(c') $\Rightarrow$ (b') This is the dual of the '(c) $\Rightarrow$ (a)' statement above; we can thus prove it in exactly the same fashion, reversing the arrows.

5.11

Since $f : A \longrightarrow B$ is flat, the induced map $f^* : \mathrm{Spec}(B_\mathfrak{q}) \longrightarrow \mathrm{Spec}(A_\mathfrak{q})$ is surjective (where $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{p}$ is an ideal of A that lies over it in A), by chapter 3, exercise 18. Therefore, by exercise 10, f has the going-down property.

5.12

It's obvious that A is integral over A^G; for, given $a \in A$, a is a root of the monic polynomial $P(x) = \prod_{\sigma\in G}(x - \sigma(a))$. We see that $P(x) \in A^G[x]$, since any automorphism of G acts as the identity on it, hence every coefficient must belong to A^G.

The obvious extension of the action of G on A to an action on $S^{-1}A$ is given by $\sigma(a/s) = \sigma(a)/\sigma(s)$, for every $\sigma \in G$. Note that this is well-defined by the $\sigma(S) \subseteq S$ condition.

Finally, given any element $a/s \in (S^{-1}A)^G$, we define a map $\phi : (S^{-1}A)^G \longrightarrow (S^G)^{-1}A^G$ by letting $a/s \mapsto \Sigma(a)/\Sigma(s)$, where $\Sigma = \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$ (we assume $G = \{\sigma_i\}_{1\leq i\leq n}$). We see that this map is well defined ($\Sigma(a\Sigma(s))$ being stable under G), and it's furthermore surjective (any element a/s gets hit my its image) and injective (because of our definition of Σ, we will have $\Sigma(a/s) = \Sigma(a)/\Sigma(s)$, whence injectivity). This completes the proof that ϕ is an isomorphism.

5.13

Let $\mathfrak{p}_1, \mathfrak{p}_2 \in P$ and let $x \in \mathfrak{p}_1$. Then,

$$\prod_{\sigma\in G} \sigma(x) \in (\mathfrak{p}_1 \cap A^G) = \mathfrak{p},$$

since $\mathrm{id} \in G$ and $\prod_\sigma \sigma(x)$ is invariant under G, hence $\sigma(x) \in \mathfrak{p}_2$ for some $\sigma \in G$. Therefore,

$$\mathfrak{p}_1 \subseteq \bigcup_{\sigma\in G} \sigma(\mathfrak{p}_2),$$

which implies that $\mathfrak{p}_1 \subseteq \sigma(\mathfrak{p}_2)$, for some $\sigma \in G$ (since the $\sigma(\mathfrak{p}_2)$ are prime). But since A is integral over A^G (by the previous exercise), and $\mathfrak{p}_1, \sigma(\mathfrak{p}_2)$ both contract to $\mathfrak{p}$, they must coincide, by (5.9). This implies that G acts faithfully, as desired.

In particular, the set of ideals that contract to $\mathfrak{p}$ is finite.

5.14

Note first of all that G is be a finite group (its order is equal to the degree of the extension L/K). It's obvious that $B \subseteq \sigma(B)$, since $\mathrm{id} \in G$. Conversely, if $b \in B$, then $\sigma(b) \in B$, since $\sigma(b) \in L$ is necessarily integral over A (since it's the identity on K, by the definition of the Galois group). Therefore, $\sigma(B) = B$.

Now, obviously $AsubseteqB^G$, and if $b \in B^G$, then b satisfies the following monic polynomial in $K[x]$:

$$\prod_{\sigma\in G} (x - \sigma(b)),$$

which implies that b is integral in K over A, hence, since the integral closure of A in K is itself, we conclude that $B^G \subseteq A$, hence the two sets are equal, as desired.

5.15

We distinguish two cases, as in the book's hint. If L is a separable extension over K, then we may embed it in a finite normal separable extension N of K. In this case, the tower $K \subseteq L \subseteq N$ of fields yields that the number of prime ideals $\mathfrak{q}$ of B which contract to $\mathfrak{p}$ in $B^G = A$ (this last equality is true by exercise 14) is finite, by exercise 13. In the case L is purely inseparable over K, then any ideal $\mathfrak{q}$ of B such that $\mathfrak{q} \cap A = \mathfrak{p}$ is in fact equal to the set $\{x \in B : x^{p^m} \in \mathfrak{p} \text{ for some } m \geq 0\}$. Since $\mathfrak{q}$ is uniquely defined in this case, we see that the induced map is bijective. Hence all fibres have one element; in particular, they are finite.

Noether's normalization lemma

5.16

We will merely repeat the hint of the book; it constitutes a full proof. For this reason, we also assume that k is infinite. Let $x_1, x_2, \ldots, x_n$ generate A as a k-algebra. By renumbering the x_i's, if necessary, we may assume that $x_1, x_2, \ldots, x_r$ are algebraically independent over k and each of the $x_{r+1}, \ldots, x_n$ are algebraic over $k[x_1, x_2, \ldots, x_r]$. Now we apply induction to the difference $n-r$; if $n = r$, then there is nothing to be shown, so assume that the proposition holds for $n-1$ generators and $n > r$. In this case, the generator x_n is algebraic over $k[x_1, \ldots, x_{n-1}]$, hence there exists a polynomial f with coefficients in k such that $f(x_1, x_2, \ldots, x_n) = 0$. We may of course assume that the homogeneous part F with the largest degree d of f is monic in the last argument (x_n), because, since k is infinite, there are $\lambda_1, \lambda_2, \ldots, \lambda_{n-1} \in k$ such that $F(\lambda_1, \lambda_2, \ldots, 1) \neq 0$, and this will be the coefficient of x_n in F so we may divide by it. Putting $x_i' = x_i - \lambda_i x_n$, we obtain from F a monic polynomial in $k[x_1', x_2', \ldots, x_{n-1}']$ that vanishes at x_n, hence $A = k[x_1, x_2, \ldots, x_n]$ is integral over $A' = k[x_1', x_2', \ldots, x_{n-1}']$. Now the inductive hypothesis yields $y_1, y_2, \ldots, y_{r-1}$ such that A' is integral over the k-algebra they generate, hence putting $y_r = x_n$, we obtain the desired result.

From the proof it now follows that we may choose the y_i to be linear combinations of the x_i. If k is algebraically closed and X is an affine algebraic variety in k^n with coordinate ring $A \neq 0$, then there exists a linear subspace L of dimension r in k^n and a linear mapping $k^n \twoheadrightarrow L$, which maps X onto L (we begin with the natural surjective map $X \twoheadrightarrow L$ and extend it using exercise 2, to a map $k^n \twoheadrightarrow L$).

Nullstellensatz, weak form

5.17

We will merely reproduce the hint of the book; it constitutes a full proof. Let, under the conditions of the problem, $A = k[t_1, t_2, \ldots, t_n]/I(X)$ and note that A is a non-empty finitely generated k-algebra. By Noether's normalization lemma (exercise 16), there is a linear subspace $L = k[y_1, y_2, \ldots, y_r]$ of dimension at least 0 in k^n, and a mapping of X onto L. In particular, $X \neq \varnothing$, as desired.

We will now derive that every maximal ideal of $A = k[t_1, t_2, \ldots, t_n]$ is of the form $\mathfrak{m} = (t_1 - a_1, t_2 - a_2, \ldots, t_n - a_n)$. Let $\mathfrak{m}$ be a maximal ideal of A and let $V_\mathfrak{m}$ be the variety it defines. Then, if $I(V_\mathfrak{m})$ is the ideal of that variety, the strong Nullstellensatz implies that $I(V_\mathfrak{m}) = r(\mathfrak{m}) = \mathfrak{m} \neq (1)$; the previous statement implies $V_\mathfrak{m} \neq \varnothing$. If $(a_1, a_2, \ldots, a_n) \in V_\mathfrak{m}$, then the previous relation implies that $\mathfrak{m} \subseteq t_1 - a_1, t_2 - a_2, \ldots, t_n - a_n)$, which is absurd, lest $\mathfrak{m} = (t_1 - a_1, t_2 - a_2, \ldots, t_n - a_n)$. This completes the proof.

5.18

Let $x_1, x_2, \ldots, x_n$ generate B as a k-algebra. We will proceed by induction on n. If $n = 1$, then the result is obvious since B is a field. Therefore assume $n > 1$ and that the result holds for $n-1$ generators. Let $A = k[x_1]$ and let $K = k(x_1)$ be the field of fractions of A. Then, by the inductive hypothesis, B is a finite algebraic extension of K, therefore, the generators $x - 2, x_3, \ldots, x_n$ satisfy monic polynomial algebraic equations in K; the coefficients of those polynomials will be of the form a/b, where $a, b \in A$. If f is the product of all the denominators b, the $x_2, x_3, \ldots, x_n$ are all clearly integral over A_f. Hence B and therefore K are integral over A_f.

Now if x_1 were algebraic over k, then, since A is a Unique Factorization Domain, A is integrally closed (if $\mathfrak{p}$ is a prime ideal in A, then unique factorization implies in particular that no prime ideals $\mathfrak{q}$ in the integral closure of A could be such that $\mathfrak{q} \cap (1) = \mathfrak{p}$, which happens by Lying Over in the case A is not closed). Hence A_f is also integrally closed and therefore $A_f = K$, an absurdity. Therefore, x_1 is algebraic over k, hence K (thus B, too) are algebraic over k. We conclude that B is a finite algebraic extension of k, as desired.

5.19

We already used the Nullstellensatz in our solution of exercise 17.

5.20

Let $S = A - \{0\}$ and let $K = S^{-1}A$ be the field of fractions of A. Since B is a finitely generated algebra over A, we may write $B = A[b_1, b_2, \ldots, b_m]$, and this implies that $S^{-1}B = K[b_1, b_2, \ldots, b_m]$ is a finitely generated algebra over K. By the Noether Normalization Lemma, there exist $y_1/s_1, y_2/s_2, \ldots, y_n/s_n \in S^{-1}B$ that are algebraically independent over K and are such that $S^{-1}B$ is integral over $K[y_1/s_1, y_2/s_2, \ldots, y_n/s_n]$. It's now easy to see that $y_1, y_2, \ldots, y_n$ are algebraically independent over K and $S^{-1}B$ is integral over $K[y_1, y_2, \ldots, y_n]$. The fact that $b_1, b_2, \ldots, b_m \in B \subseteq S^{-1}B$ are integral over $K[y_1 y_2, \ldots, y_n]$ implies that there are equations

$$b_i^{r_i} + \frac{a_{(i,1)}}{s} + \cdots + \frac{a_{(i,r_i)}}{s} = 0,$$

where $a_{(i,j)} \in B' = A[y_1, y_2, \ldots, y_n]$ and $s \in S$ for all $1 \leq i \leq m$ (note that we can reduce all the denominators to the same $s \in S$). Clearly, the above equations imply that $b_1, b_2, \ldots, b_m$ are all integral over B'_s. We conclude that B_s is integral over B'_s, as desired.

5.21

With the notation of exercise 20, we may first extend f to all of B', e.g. by sending $y_i \mapsto 0$, then to B'_s, since $f(s) \neq 0$ and finally to B_s, by exercise 2.

5.22

We will merely repeat the book's hint; it constitutes a full proof. Assuming that $\mathfrak{J}_A = 0$ and given any $v \in B$, we wish to construct a maximal ideal of B that doesn't contain v. By applying exercise 21 to the ring B_v and its subring A (we use the usual embedding $A \longrightarrow B_v$ and this is injective, because A is an integral domain) we obtain an $s \in A - \{0\}$ such that, if Ω is an algebraically closed field and $f : A \longrightarrow \Omega$ doesn't vanish at s, then f can be extended to a homomorphism $B \longrightarrow \Omega$. Let $\mathfrak{m}$ be a maximal ideal of A such that $s \notin A$, and let $k = A/\mathfrak{m}$ be the residue field. Then, the canonical projection $A \longrightarrow k$ extends to a homomorphism $g : B_v \longrightarrow \Omega$, where Ω is the algebraic closure of k (this follows, of course, by the was $s \neq 0$ was chosen). By proposition 5.23, $g(v) \neq 0$ and $\mathfrak{n} = \ker(g) \cap B$ is necessarily a maximal ideal in B (since $\ker(g)$ is maximal and g^* is always a continuous map). The ideal $\mathfrak{n}$ we constructed doesn't contain v, and this completes the proof.

5.23

We have the following:

(i) $\Rightarrow$ (ii) Let $f : A \longrightarrow B$ be a surjective homomorphism; then all the prime ideals of B will be given by intersections of maximal ideals of (since $f^* : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is continuous and maximal ideals correspond to closed sets). This implies that the nilradical of B is contained in its Jacobson radical, and the converse is always true. Thus $\mathfrak{R}_B = \mathfrak{J}_B$ in every homomorphic image of A.

(ii) $\Rightarrow$ (iii) Let $\mathfrak{p}$ be any prime but not maximal ideal of A and let $B = A/\mathfrak{p}$ (which is of course a homomorphic image of A under the natural projection). Then, the Jacobson radical of B will equal the nilradical of B, which is 0. Since the Jacobson radical of B is merely the intersection of the images of all maximal ideals of A that contain $\mathfrak{p}$ strictly, we may pull back to A, to obtain that $\mathfrak{p}$ is equal to the intersection of all prime ideals that contain it strictly (we substituted *maximal* by *prime* because we are considering only the intersection of the ideals involved).

(iii) $\Rightarrow$ (i) Assume that there is a prime ideal $\mathfrak{p}$ of A, which is not the intersection of maximal ideals, and let $B = A/\mathfrak{p}$. Then, B is an integral domain and its zero ideal is not the intersection of maximal ideals; in particular, its Jacobson radical is nonzero. Pick $f \neq 0$ a non-zero element of $\mathfrak{J}_B$. Then, B_f is a non-zero ring, since f is nonzero and if $\mathfrak{m}$ is any maximal ideal of B_f, then $\mathfrak{q} = \mathfrak{m} \cap B$ is a prime ideal of B, which, however, is not maximal (otherwise f would be an element of $\mathfrak{q}$). Moreover, by this construction, every prime ideal of B that meets $\mathfrak{q}$, also meets the multiplicatively closed set $\{f^n\}_{n \geq 0}$, and so contains f. This mean that f is contained in the intersection of all ideals that contain $\mathfrak{q}$ strictly. But condition (iii) implies that $\mathfrak{q}$ is equal to the intersection of all maximal ideals that contain it strictly, a contradiction, as desired.

5.24

We have the following:

(i) If B is integral over A, then passing to $A/\mathfrak{R}_A$, we may assume that $\mathfrak{R}_A = \mathfrak{J}_A = 0$. Now, by integrality, $\mathfrak{J}_B = A \cap \mathfrak{J}_A$, hence $\mathfrak{R}_B \subseteq \mathfrak{J}_B = 0$ implies $\mathfrak{J}_B = \mathfrak{R}_B$, or that B is a Jacobson ring, as desired.

(ii) If B is a finitely generated A-algebra, then by passing to $A/\mathfrak{R}_A$, where $\mathfrak{R}_A$ is the nilradical of A, we may assume that $\mathfrak{R}_A = \mathfrak{J}_A = 0$. Then, exercise 22 yields that $\mathfrak{R}_B \subseteq \mathfrak{J}_B = 0$, hence B is Jacobson too, as desired.

In particular, every finitely generated ring and every finitely generated algebra over a field is a Jacobson ring.

5.25

We have the following:

(i) $\Rightarrow$ (ii) Without loss of generality, we may assume that A is a subring of B (since if $B \cap A \subseteq A$ is finite over A, then so will B be, since as a field it is a finite algebraic extension of A). Applying exercise 21 yields some $s \in A$ such that if $f : A \longrightarrow \Omega$ is a homomorphism which doesn't vanish at s, then f can be extended to a homomorphism $B \longrightarrow \Omega$. There exists a maximal ideal $\mathfrak{m}$ of A such that $s \notin \mathfrak{m}$ and the canonical projection $A \longrightarrow A/\mathfrak{m} = k$ extends to a map $g : B \longrightarrow \Omega$, where Ω is the algebraic closure of k. Since B is a field, g is injective and $g(B)$ is algebraic over k, hence finite algebraic over k, by the Nullstellensatz. But $B \simeq g(B)$, which completes the proof.

(ii) $\Rightarrow$ (i) Let $\mathfrak{p}$ be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let f be a non-zero element of B; then, B_f is a finitely generated A-algebra (since B is). If it were a field, then it would be finite over B, hence integral over B, hence B would be a field by the Nullstellensatz, a contradiction. Thus B_f is not a field and therefore it has some proper prime ideal $\mathfrak{p}$. It's contraction in B is a non-zero ideal $\mathfrak{p}'$ such that $f \notin \mathfrak{p}'$. Since $f \in A - \mathfrak{p}$ was arbitrary, we conclude that the intersection of all maximal ideals that contain $\mathfrak{p}$ is contained in $\mathfrak{p}$ and this obviously implies the equality of the two sets.

5.26

First we prove the topological equivalences:

(1) $\Rightarrow$ (2) Since E is closed, $\overline{E \cap X_0} \subseteq E$. Conversely, if $x \in E$, then any open neighborhood U of x meets $E \cap X_0$ (it meets E at x and since $U = U \cap \{x\}$, it also meets X_0). This implies $x \in \overline{E \cap X_0}$, hence the desired equality.

(2) $\Rightarrow$ (3) The map $U \longrightarrow U \cap X_0$ is surjective, by the definition of the subspace topology, and it's surjective, because $U_1 \cap X_0 = U_2 \cap X_0$ implies $U_1 - U_2 = \overline{(U_1 - U_2) \cap X_0} = \varnothing$.

(3) $\Rightarrow$ (1) If the above map is bijective, then we easily see that for every proper open (resp. closed) set G (resp. F) $G \cap X_0 \neq \varnothing, X$ (resp. $F \cap X_0 \neq \varnothing, X$). Therefore, if $L = F \cap G$ is any locally closed subset of X, then the proper subspace $U \cap X_0$ of U, locally satisfies condition (3), therefore in particular the intersection of the set $(F \cap U)$, which is closed in U, with $U \cap X_0$ is non-empty. But then we have $\varnothing \neq (F \cap U) \cap (U \cap X_0) = L$, and this yields the desired result.

A subset X_0 of X that fulfills these conditions is called *very dense*.

For the algebraic equivalences, we have:

(i) $\Rightarrow$ (ii) Let L be any locally closed set in $\mathrm{Spec}(A)$. This can be precisely characterized as the set of prime ideals of A that contain some fixed ideal α of A but do not contain some fixed $a \in A$. Passing on to A/α, we may assume that $\alpha = \mathbf{0}$. Since $A \neq \varnothing$ (this is where the condition $f \notin \mathfrak{p}$ for all $\mathfrak{p} \in L$ comes in), A will have a maximal ideal $\mathfrak{m}$, Pulling back to our original A, we see that $\mathfrak{m} \in L$ and this implies that the set of all maximal ideals of A is very dense in $\mathrm{Spec}(A)$.

(ii) $\Rightarrow$ (iii) This direction is obvious, since the single point-set must meet the set of all maximal ideals, which are of course the closed points in $\mathrm{Spec}(A)$.

(iii) $\Rightarrow$ (i) Let $\mathfrak{p}$ be any prime ideal which is not maximal; this will correspond to a point in $\mathrm{Spec}(A)$, which, as a set, is not closed. The intersection of all maximal ideals of A that strictly contain $\mathfrak{p}$ also contains $\mathfrak{p}$. If the converse were not true, then there would be a non-empty set $F \subseteq A$ such that F is contained in any maximal ideal $\mathfrak{m}$ of A which contains $\mathfrak{p}$, but F is not contained in $\mathfrak{p}$. But then, $\{\mathfrak{p}\}$ in $\mathrm{Spec}(A)$ would be equal to the locally closed set $X_F \cap \mathfrak{p}$, which, since it consists of one point only, should be closed, an absurdity. This shows that A is Jacobson, as desired.

Valuation rings and valuations

5.27

Since intersection of sets is associative, a typical Zorn's lemma argument shows that Σ indeed has maximal elements with respect to domination; let A be one of them. We claim that A is a valuation ring of K and conversely. Indeed, if A is a ring maximal with respect to domination, then A is maximal in the sense of theorem 5.21, if we let (A, f) be the pair of A together with its embedding in $K = \Omega$. Conversely, if A is a valuation ring, then it's maximal in the sense described above; we claim that it's also maximal with respect to domination. Indeed, if $A \supseteq B$ and $\mathfrak{m}_A \subseteq \mathfrak{m}_B$, then, since $\mathfrak{m}_B \subseteq \mathfrak{m}_A$, we must have $\mathfrak{m}_A = \mathfrak{m}_B$. This, however implies that all the elements of $B - A$ are units in B but not in A with their inverses in B but not in A; this contradicts the fact that A is a valuation ring of $K \supseteq B \supseteq A$, hence if a $k \in B$, then $k \in A$ or $k^{-1} \in A$.

5.28

We have the following:

(1) $\Rightarrow$ (2) Let $\mathfrak{a}, \mathfrak{b}$ be two proper ideals of the valuation ring $A \subseteq K$. If none of them were included in the other, then there would be $\alpha \in \mathfrak{a}, \beta \in \mathfrak{b}$ such that $\alpha \notin \mathfrak{b}, \beta \notin \mathfrak{a}$. Since A is a local ring, we will have $k = \alpha\beta^{-1} \in A$ or $k' = \alpha^{-1}\beta \in A$; assume without loss of generality the former. Then, $\alpha = k\beta \in \mathfrak{b}$, a contradiction.

(2) $\Rightarrow$ (1) Consider the set $\{\mathfrak{a} \cup \{0\}\}$, where $\mathfrak{a}$ runs over all the ideals of A, of subrings of K; this will be totally ordered by set inclusion $\subseteq$ and the inclusion maps $\mathfrak{a} \cup \{0\} \longrightarrow K$ yield a set Σ analogous to the one constructed in page 65. A maximal element of that is clearly A itself, and therefore, by theorem 5.21, A will be a valuation ring of K.

We deduce that if $\mathfrak{p}$ is a prime ideal of A, then, $A_\mathfrak{p}$ and $A/\mathfrak{p}$ are valuation rings of their respective fields of fractions, since the second condition continues to hold after modding out by $\mathfrak{p}$ or localizing at $\mathfrak{p}$.

5.29

We will assume that the phrase $A \subseteq B$ *is a local ring of* B means: A *is a local ring and so is* B *and their unique maximal ideals coincide*. Now, with the notation of the problem, let $A \subseteq B \subseteq K$ be an intermediate ring. Since A is a valuation ring of K (hence it's local, with maximal ideal $\mathfrak{m}_A$), B will be a local valuation ring of K, too, by proposition 5.18; let $\mathfrak{m}_B$ be its unique maximal ideal. Obviously $\mathfrak{m}_A \subseteq \mathfrak{m}_B$ (by the valuation conditions) and if there were $x \in \mathfrak{m}_B - \mathfrak{m}_A$, then x would have to be a unit in A, but a non-unit in B, a contradiction. Thus $\mathfrak{m}_A = \mathfrak{m}_B$, as desired.

5.30

We first note that the relations defined induce an ordering of Γ; indeed, $\xi \geq \xi$, $\xi \geq \eta, \eta \geq \xi$ implies that $\xi^{-1}\eta \in U$ and therefore $\xi = \eta$, and finally $\xi \geq \eta, \eta \geq \omega$ obviously implies $\xi \geq \omega$. Since A is also a valuation ring, the ordering described above is also total, in the sense that for any $\xi, \eta \in \Gamma$, $\xi \geq \eta$ or $\eta \geq \xi$. This structure is compatible with the group structure, since $\xi \geq \eta \Rightarrow \xi\omega \geq \eta\omega$, for all $\omega \in \Gamma$. Hence Γ is a totally ordered abelian group, the *value group* of A.

Letting $v : K^* \longrightarrow \Gamma$ be the natural projection homomorphism, and keeping the previous notation, we would like to show that $(x+y)y^{-1} \in A$ or $(x+y)x^{-1} \in A$; this just follows from the definition of valuation rings.

5.31

We will rather consider the set $A = \{x \in K^* : v(x) \geq 0\} \cup \{0\}$, so that A is actually a ring. Conditions (1) and (2) imply that A is closed with respect to multiplication and addition and A is an integral domain, since all its elements belong to the field K. Note that $x = y = 1$ in (1), yields $v(1) = 0$, therefore, $v(x) = v(x^{-1})$ for all $x \in K$. This implies that $if v(x) \geq 0$ or $v(x) \leq 0$, thus $x \in A$ or $x^{-1} \in A$. Hence A is a valuation ring (*the valuation ring of* v) of K. The subgroup $v(K^*)$ of Γ is the *value group of* v.

5.32

It's obvious that $v(A - \mathfrak{p}) = \{v(a) : a \notin \mathfrak{p}\}$, and this is an isolated component Δ of Γ (this is where the primality of $\mathfrak{p}$ comes in). The mapping from $\mathrm{Spec}(A)$ into the set of isolated components of Γ is obviously injective; the surjectivity follows from the inverse construction of exercise 30.

5.33

The group algebra A is an integral domain, because writing two non-zero elements as below $u = \lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n}, u' = \lambda'_1 x_{\alpha'_1} + \cdots + \lambda'_n x_{\alpha'_n}$ and multiplying them yields that the two terms of lowest degree will yield the term of lowest degree in the product uu'. Since k is an integral domain, the coefficient of that term will be non-zero, hence so will the whole product be. This shows that the group algebra $k[\Gamma]$ over an integral domain is always an integral domain.

If we write an arbitrary $u \in k[\Gamma]$ in the form $u = \lambda_1 x_{\alpha_1} + \cdots + \lambda_n x_{\alpha_n}$, where $\lambda_i \in k - \{0\}$ and $\alpha_i < \alpha_{i+1}$ for all i, and then define a mapping $v : A - \{0\} \longrightarrow \Gamma$ by letting $u(v) = \alpha_1$, we see that:

(a) v is a homomorphism, namely $v(xy) = v(x) + v(y)$.

(b) $v(x+y) \geq \min(v(x), v(y))$, as desired.

Now, if K is the field of fractions of A, then we may uniquely extend it to $K = (A - \{0\})^{-1}A$, by letting $v(a/s) = v(a)/v(s)$, if $a \neq 0$ and of course letting $v(0/1) = \infty$. The value group of this valuation will be Γ.

5.34

If we follows the hint of the book and put $C = g(B)$, then it's obvious that $C \supseteq g(A)$, since $a = g(f(a))$, for all $a \in A$. For the other direction, let $\mathfrak{n}$ be a maximal ideal of C; since f^* is a closed mapping, the restriction of $\mathfrak{n}$ is the maximal ideal $\mathfrak{m}$ of A (equivalently, $\mathfrak{m} = A \cap \mathfrak{n}$). This implies that $A_\mathfrak{m} = A$ (since the elements of the multiplicative set $A - A \cap \mathfrak{n}$ are the units). By the given relations, $C_\mathfrak{n}$ dominates $A_\mathfrak{m} = A$, but since A is a valuation ring, it will be maximal with respect to domination, therefore, by exercise 27, $C_\mathfrak{m} = A$ or $C \subseteq A$. This completes the proof that $A = C$.

5.35

We will largely repeat the hint of the book; it constitutes a full proof. From exercises 1-3 it follows that, if $f : A \longrightarrow B$ is integral and C is an A-algebra, then $(f \otimes_A 1)^* : \mathrm{Spec}(B \otimes_A C) \longrightarrow \mathrm{Spec}(C)$ is a closed map.

Conversely, suppose that f has this property and that B is an integral domain; then f is integral. Replacing A by its image in B, we can assume without loss of generality that $A \subseteq B$ and f is merely inclusion $A \longrightarrow B$. Let K be the field of fractions of B and let A' be a valuation ring containing A (there is at least one, since A is contained in its integral closure and this is precisely the intersection of all valuation rings that contain A; in particular, there is at least one such valuation ring). By corollary (5.22), it is enough to show that $B \subseteq A'$. By the hypothesis, $\mathrm{Spec}(B \otimes_A A') \longrightarrow \mathrm{Spec}(A')$ is a closed map. Applying the result of exercise 34 to the homomorphism $B \otimes_A A' \longrightarrow K$, defined by $b \otimes_A a' \mapsto ba'$ yields that $ba' \in A'$ for all $b \in B, a' \in A'$. In particular, for $a' = 1$, we obtain $B \subseteq A'$.

This result remains valid if B is any ring with a finite number of minimal prime ideals (e.g if B is Noetherian). Indeed, if $\{\mathfrak{p}_i\}_{1 \leq i \leq n}$ are the minimal prime ideals, then each composite homomorphism $A \longrightarrow B \longrightarrow B/\mathfrak{p}_i$ is integral, hence so is the induced map $A \longrightarrow \prod(B/\mathfrak{p}_i)$, hence so is $A \longrightarrow A/\mathfrak{R}$, and finally so is $A \longrightarrow B$. This completes the proof.

Chapter 6

Chain Conditions

6.1

We have the following:

(i) Assume that u is not injective, namely that there is a non-zero $x \in \ker u$. Since u is surjective, there is $y \in M$, such that $u(y) = x$ and obviously y does not belong to $\ker u$; in particular $y \neq x$. Therefore the inclusion $\ker u \subset \ker u^2$ is strict and similarly the inclusion $\ker u^n \subset \ker u^{n+1}$ is strict for all $n \in \mathbb{N}$. If we let K_n be the kernel of u^n, then $\{K_n\}_{n\in\mathbb{N}}$ is an ascending chain of submodules of M without a maximal element, which is absurd by the Noether condition on M. Therefore u is injective and an isomorphism.

(ii) An identical argument for the cokernels of u^n shows that u is surjective (and thus an isomorphism) in case M is Artinian.

6.2

If M was not Noetherian, then there would be a non-finitely generated submodule N of M. Let $x_1 \in N$. Since N is not finitely generated, $N - (x_1) \neq \varnothing$, hence there is $x_2 \in N - (x_1)$. For the same reason, there is $x_3 \in N - (x_1, x_2)$ and in this fashion we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of M such that the non-empty set of finitely generated submodules $\{(x_1), (x_1, x_2), (x_1, x_2, x_3), \ldots\}$ has no maximal element, a contradiction. Therefore, M is Noetherian.

6.3

Under the conditions of the problem, we have the following exact sequence

$$0 \longrightarrow M/N_1 \xrightarrow{incl} M/(N_1 \cap N_2) \xrightarrow{\pi} M/N_2 \longrightarrow 0,$$

where *incl* and π are the natural inclusion and projection maps respectively. By Proposition 6.3, $M/(N_1 \cap N_2)$ is Noetherian (resp. Artinian) if and only if M/N_1 and M/N_2 are Noetherian (resp. Artinian).

6.4

As a Noetherian A-module, M will be finitely generated, say by $x_1, x_2, \ldots, x_n$. Let

$$f : A \longrightarrow \bigoplus_{i=1}^{n} M = M^n$$

send a to $(ax_1, ax_2, \ldots, ax_n) \in M^n$. Note that $\ker f = \mathfrak{a}$ (since $f(a) = 0$ if and only if $ax_1 = ax_2 = \ldots = ax_n = 0$ therefore if and only if $aM = 0$ which is equivalent to $a \in \mathfrak{a}$). But then $A/\ker f$ can be embedded isomorphically in M^n, which is itself a Noetherian module (by corollary 6.4). As a submodule of a Noetherian module, $A/\mathfrak{a}$ will be Noetherian itself, as desired.

The statement collapses if we replace the Noetherian condition by the Artinian one. For example, let $A = \mathbb{Z}$ and $M = G$ be the subgroup of $\mathbb{Q} - \mathbb{Z}$ consisting of all elements with absolute value equal to $1/p^n$, $n \geq 0$ (of course p is some fixed prime number, as in the book's example). As shown in the book, G is an Artinian $\mathbb{Z}$-module, and its annihilator obviously equals 0. If the previous statement was still true in the case of Artinian modules, then it would yield that $\mathbb{Z} = A/\mathrm{Ann}(M)$ is Artinian, which is false.

6.5

If Y is a subspace of X with the induced topology, then any set U open in Y is of the form $G \cap Y$, where G is a set open in X. Therefore, any ascending chain $\{U_n\}_{n\in\mathbb{N}}$ of sets open in Y is is of the form $\{G_n \cup Y\}_{n\in\mathbb{N}}$, where $\{G_n\}_{n\in\mathbb{N}}$ is an ascending chain of sets open in X. Since X is Noetherian, $\{G_n\}_{n\in\mathbb{N}}$ is stationary, hence so will $\{U_n\}_{n\in\mathbb{N}}$ be. Therefore, any subspace of X is Noetherian.

Assume that X is Noetherian, but not quasi-compact (the definition of the book seems to be that of usual compactness). Then, there is an open cover $\mathfrak{C} = \{C_i\}_{i\in I}$ of X, such that no finite subcover of $\mathfrak{C}$ covers X. Let G_1 be an arbitrary element of $\mathfrak{C}$. Since G_1 doesn't cover X, there is $x \in X$ such that $x \notin G_1$, but $x \in C_i$ for some $i \in I$; let $G_2 = G_1 \cup C_2$. Since G_2 fails to cover X, there is $x' \in C_j \subset X$ with $x' \notin G_2$; let $G_3 = G_1 \cup G_2 \cup C_j$. In this fashion, we construct an ascending chain of open sets $\{G_n\}_{n\in\mathbb{N}}$ that is not stationary, contrary to the assumption that X is Noetherian. Therefore, X is quasi-compact.

6.6

We will follow an unorthodox order in the proof of the equivalences:

(i) $\Rightarrow$ (iii) By the previous exercise, every subspace of X will be Noetherian and thus quasi-compact (by the previous exercise again).

(iii) $\Rightarrow$ (ii) O.K.

(ii) $\Rightarrow$ (i) Let $\mathfrak{C} = \{G_n\}_{n\in\mathbb{N}}$ be an ascending chain of open subsets of X. Then, since the subspace

$$Y = \bigcup_{n\in\mathbb{N}} G_n$$

of X is quasi-compact and has $\mathfrak{C}$ as an open cover, there will be a finite subcover $\{G_{i_n}\}_{1\leq n\leq N}$ of Y. But, if $i = \max\{i_1, i_2, \dots, i_N\}$, then

$$Y = \bigcup_{1\leq n\leq N} G_{i_n} = G_i.$$

This G_i is clearly an upper bound of $\mathfrak{C} = \{G_n\}_{n\in\mathbb{N}}$, which shows that X is Noetherian.

6.7

We know that the maximal irreducible subspaces $\{Y_i\}_{i\in I}$ are closed and cover X, by chapter 1, exercise 20, (iii).

If we assume that the intersection of all the Y_i is non-empty (let x be some point of X contained in it), then any neighborhood of any point of X intersects any neighborhood of x, therefore we conclude that $X = \overline{\{x\}}$. In this case the statement is vacuously true.

In the case $\cap_{i\in I} Y_i = \varnothing$ the set $\{X - Y_i\}_{i\in I}$ is an open cover of X and this necessarily has a finite subcover (since X is Noetherian and thus quasi-compact) and this in turn yields a presentation of X as a finite union of irreducible and closed maximal subspaces.

We deduce that the set of irreducible components of a Noetherian space (which are $V(\mathfrak{p}_\mathfrak{i})$, $i \in I$ and the $\mathfrak{pi}$ are the minimal prime ideals of A) is finite.

6.8

Assume that $\{V(\mathfrak{a}_n)\}_{n\in\mathbb{N}}$ is a decreasing sequence of closed sets in $\operatorname{Spec}(A)$ (the $\mathfrak{a}_n$ are assumed radicals of ideals of A; this assumption leads to no loss of generality, since $V(\mathfrak{a}) = V(r(\mathfrak{a}))$). Since V is an inclusion-reversing bijection from the set of all radicals of A to the set of all closed subsets of $\operatorname{Spec}(A)$, this chain yields an increasing chain $\{\mathfrak{a}_n\}_{n\in\mathbb{N}}$ which must stabilize by the Noether condition on A. This implies that the initial chain stabilizes too and therefore $\operatorname{Spec}(A)$ is a Noetherian topological space.

The converse, however, is not true. Let $A = k[x_1, x_2, \ldots]$ be a polynomial ring over a ring A with infinitely many variables adjoined and let $\mathfrak{a} = (x^1, x^2, \ldots)$ be the ideal generated by the squares of all the variables. Then, $\tilde{A} = A/\mathfrak{a}$ is not Noetherian (because A is not), but note that $\mathfrak{R}(\tilde{A}) = (\overline{x}_1, \overline{x}_1, \ldots)$ thus $\operatorname{Spec}(\tilde{A}) = \operatorname{Spec}(\tilde{A}/\mathfrak{R})$ (by chapter 1, exercise 21, (iv)) and this is a point, hence it's trivially Noetherian.

6.9

If A is a Noetherian ring, then $\operatorname{Spec}(A)$ is a Noetherian topological space (with the standard Zariski topology) by exercise 8. Therefore, by exercise 7, $\operatorname{Spec}(A)$ will have a finite number of irreducible subspaces, which are exactly the minimal prime ideals of A by chapter 1, exercise 20, (iv).

6.10

$\operatorname{Supp}(M)$ is closed since it equals $V(\operatorname{Ann}(M))$ by chapter 3, exercise 19, (v). Now let $\{\operatorname{Supp}(M)\cap V(\mathfrak{a}_i)\}_{i\in I}$ be a decreasing sequence of closed subsets of $\operatorname{Supp}(M)$. By chapter 3, exercise 19, we see that $\operatorname{Supp}(M)\cap\mathfrak{a}_i = V(\operatorname{Ann}(M)\cap V(\mathfrak{a}_i) = V(\operatorname{Ann}(M)\cup V(\mathfrak{a}_i) = V(\mathfrak{b}_i)$ (where $\mathfrak{b}_i$ is the ideal generated by $\operatorname{Ann}(M)\cap V(\mathfrak{a}_i)$). We may assume that $\mathfrak{b}_i$ is a radical, since $V(\mathfrak{e}) = V(r(\mathfrak{e}))$ for all ideals $\mathfrak{e}$ of A. Since V is an inclusion-reversing bijection between the radicals of A and the closed subsets of $\operatorname{Spec}(A)$, we obtain an increasing sequence $\{\mathfrak{b}_i\}_{i\in I}$ of ideals of A. Letting $\mathfrak{p}_i$ be the minimal prime ideal that contains $\mathfrak{b}_i$ gives rise to an increasing sequence $M_{\mathfrak{p}_i}$ of submodules of M, which stabilizes because M is Noetherian. Therefore the initial sequence must stabilize too and this completes the proof that $\operatorname{Supp}(M)$ is a Noetherian topological space.

6.11

By chapter 5, exercise 10, (i) $f^* : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ always has the going-up property if it's a closed mapping.

If $\operatorname{Spec}(B)$ is a Noetherian space, then the converse is also true. Indeed, let $V(\mathfrak{p}) \subseteq \operatorname{Spec}(B)$; then, by the equivalent condition (c) of chapter 5, exercise 10, we have that $V(\mathfrak{q}) \subseteq f^*(V(\mathfrak{p}))$, where $\mathfrak{q}$ is merely the restriction of $\mathfrak{p}$ in A. Then, we would like to show that the map is also injective, so that the closed set $V(\mathfrak{p})$ is mapped to a closed set. If the inclusion was strict, then we would have the infinite strictly descending chain (the $\mathfrak{p}_i$ arise from the going-up property):

$$f^{*-1}(V(\mathfrak{q})) \supsetneq V(\mathfrak{p}) \supsetneq V(\mathfrak{p}_1) \supsetneq V(\mathfrak{p}_2) \supsetneq \ldots$$

of closed sets in $\operatorname{Spec}(B)$ (note that f^* is always continuous, hence the preimage of any closed set is closed), a contradiction by the fact that $\operatorname{Spec}(B)$ is Noetherian. Therefore f^* is closed and this completes the proof.

6.12

Before the actual proof, note that given ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A, $V(\mathfrak{p}) = V(\mathfrak{p})$ if and only if $\mathfrak{p} = r(\mathfrak{p}) = r(\mathfrak{q}) = \mathfrak{q}$.

Now, if $\{\mathfrak{p}_n\}_{n\in\mathbb{N}}$ is an ascending chain of prime ideals in A, then $\{V(\mathfrak{p}_n)\}_{n\in\mathbb{N}}$ is a descending chain of closed sets in $\operatorname{Spec}(A)$ and since this space is assumed Noetherian, the sequence $\{V(\mathfrak{p}_n)\}_{n\in\mathbb{N}}$ must be stationary, hence so will the sequence $\{\mathfrak{p}_n\}_{n\in\mathbb{N}}$ be.

The converse follows in the same fashion, since $V(\mathfrak{p}) \supseteq V(\mathfrak{q}) \Leftrightarrow \mathfrak{p} \subseteq \mathfrak{q}$ for any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of A.

Chapter 7

Noetherian Rings

7.1

We note that Σ has maximal ideals and this follows by a typical Zorn's Lemma argument. Given such a maximal ideal $\mathfrak{a}$, assume that $x, y \notin \mathfrak{a}$, but $xy \in \mathfrak{a}$. Then, $\mathfrak{b} = \mathfrak{a} + (x)$ strictly contains $\mathfrak{a}$, therefore it must be finitely generated; say $\mathfrak{b} = \mathfrak{a}_0 + (x)$, where $\mathfrak{a}_0$ is also finitely generated. Note that $\mathfrak{a} + (x) = \mathfrak{a}_0 + (x)$ implies $\mathfrak{a}_0 \subseteq \mathfrak{a}$. We claim it also implies $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x)$. Indeed, the right hand side is obviously contained in the left hand side and conversely, given any $a \in \mathfrak{a}$, $a + xt \in \mathfrak{a}_0 + (x)$ for every $t \in A$. But then, there are $a_0 \in \mathfrak{a}_0, k \in A$ such that $a = a_0 + x(k - t)$, and, since $x(k - t) \in \mathfrak{a}$, $k - t \in (\mathfrak{a} : x)$, therefore the left hand side is contained in the right hand side too. This shows that the desired equality holds. Since $(\mathfrak{a} : x)$ strictly contains $\mathfrak{a}$ it must be finitely generated, therefore $\mathfrak{a} = \mathfrak{a}_0 + x(\mathfrak{a} : x)$ is also finitely generated, a contradiction. Thus $\mathfrak{a}$ is prime.

As a corollary to the above, we observe that a ring in which every prime ideal is finitely generated is Noetherian (I.S. Cohen).

7.2

Assume that $f = a_0 + a_1x + a_2x^2 + \cdots \in A[[x]]$ is nilpotent; we will show that all its coefficients a_n, $n \geq 0$ are nilpotent. For that, we just need to show that given any prime ideal $\mathfrak{p}$ and any coefficient a_n, $a_n \in \mathfrak{p}$. Indeed, let $\overline{A} = A/\mathfrak{p}$ be the ring obtained by reduction of A modulo $\mathfrak{p}$; since $\overline{A}$ is an integral domain, so will $\overline{A}[[x]]$ be (this is trivial to check). In particular, $\overline{A}[[x]]$ will contain no nonzero nilpotent elements. The natural projection $A \twoheadrightarrow \overline{A}$ lifts naturally to a projection $A[[x]] \twoheadrightarrow \overline{A}[[x]]$ and since f is nilpotent, $\overline{f}$ will also be nilpotent, which yields $\overline{f} = \overline{0}$ by the above. Hence every coefficient a_n of f gets mapped to $\overline{0} = 0 + \mathfrak{p}$ in $\overline{A}$, which implies $a_n \in \mathfrak{p}$, as desired.

Conversely, if all the coefficients of $f \in A[[x]]$ are nilpotent, and A is a Noetherian ring, then f is also necessarily nilpotent. By the Noetherian condition on A, there is a positive integer k such that $\mathfrak{R}^k = 0$, where $\mathfrak{R}$ is the nilradical of A. We easily then see that $f^k = 0$, hence f is nilpotent.

7.3

We have the following:

(i) $\Rightarrow$ (ii) If $\mathfrak{a} \cap S = \varnothing$, then the restriction of $S^{-1}\mathfrak{a} = \mathfrak{a}$, hence we may just put $x = 1$. Otherwise, $(S^{-1}\mathfrak{a})^c = A$, thus $x = 0$ satisfies the given condition.

(ii) $\Rightarrow$ (iii) Assume otherwise; in particular, if $S = \{x^n\}_{n \geq 0}$, then $S \cap \mathfrak{a} = \varnothing$. Therefore, $(\mathfrak{a} : y) = (S^{-1}\mathfrak{a})^c = \mathfrak{a}$ for some $y = x^m$. But then, $x^{km} = y^k = 1$ for some $k \in \mathbb{N}$, and the sequence terminates, which is absurd.

(iii) $\Rightarrow$ (i) We may assume that $\mathfrak{a} = 0$, passing on to $A/\mathfrak{a}$ if necessary, and then we may repeat the proof of lemma 7.12 since all the chains appearing in that proof are annihilators, and $\operatorname{Ann}(z) = (0 : z)$, for all $z \in A$.

7.4

We have the following:

(i) This ring is isomorphic to the ring of rational functions $\mathbb{C}(t)$, hence it's not Noetherian.
(ii) This set is not even a ring (it doesn't contain the **0** power series).
(iii) This ring is isomorphic to the ring of all rational functions $\mathbb{C}(t)$, hence it's not Noetherian.
(iv) This ring is isomorphic to $\mathbb{C}[z]$, hence it's Noetherian.
(v) This ring is isomorphic to $\mathbb{C}[z, w]$, hence it's Noetherian.

7.5

Since B is integral over B^G (any $x \in B$ is a root of $\prod_{\sigma\in\Sigma}(x - \sigma(x)) \in B^G[x]$), proposition 7.4 implies that B^G is a finitely generated A-generated algebra.

7.6

If the characteristic of K is 0, then $\mathbb{Z} \subset \mathbb{Q} \subseteq K$, and since K is finitely generated over $\mathbb{Z}$, it will be so over $\mathbb{Q}$, therefore proposition 7.8 yields that $\mathbb{Q}$ is finitely generated over $\mathbb{Z}$, an absurdity. Therefore, the characteristic of K equals some prime p and K is a finitely generated algebra over $\mathbb{Z}/p\mathbb{Z}$, which, by the Nullstellensatz, implies that K is a field. In particular, it's a finite field, as desired.

7.7

An immediate corollary of the Nullstellensatz is that if h is a polynomial that vanishes everywhere an irreducible polynomial p does, then $p|h$. This fact implies that the variety X is well defined as the zero locus of the irreducible polynomials $\{f_{\alpha_i}\}_{\alpha\in I_0}$ and by the same argument there cannot be more then d such polynomials, where d is the minimal of their degrees. In particular, the variety is defined by a finite number of polynomials.

7.8

Indeed, if $A[x]$ is Noetherian, then so is A. Let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_n \subseteq \ldots$ be any ascending chain of ideals in A. This induces an ascending chain $\mathfrak{a}_1[x] \subseteq \mathfrak{a}_2[x] \subseteq \cdots \subseteq \mathfrak{a}_n[x] \subseteq \ldots$ in $A[x]$ (the $\mathfrak{a}_i[x]$ have the obvious meaning here). This chain stabilizes, by the Noetherian condition on $A[x]$; say $mathfraka_n = \mathfrak{a}_{n+1} = \ldots$. Then, if there were $y \in \mathfrak{a}_{n+1}$ such that $y \notin \mathfrak{a}_n$, we would have $y \in \mathfrak{a}_{n+1}$ (the constant polynomial equal to that value), but $y \notin \mathfrak{a}_n$, a contradiction. This completes the proof.

7.9

Let $\mathfrak{a}$ be a n ideal of A and let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$ be the maximal ideals that contain $\mathfrak{a}$. Let x_0 be a non-zero element of $\mathfrak{a}$ and let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots \mathfrak{m}_{r+s}$ be the maximal ideals that contain x_0. Since $\mathfrak{m}_{r+1}, \mathfrak{r}+\mathfrak{2}, \ldots, \mathfrak{m}_{r+s}$ do not contain $\mathfrak{a}$, there exist $x_j \in \mathfrak{a}$, $1 \leq j \leq s$, such that $x_j \notin \mathfrak{m}_j$. Since each $A_{\mathfrak{m}_j}$ is Noetherian, the extension $A_{\mathfrak{m}_j}\mathfrak{a}$ of $\mathfrak{a}$ in $A_{\mathfrak{m}_j}$ is finitely generated; let $x_i, x_2, \ldots, x_t$ be the elements of $\mathfrak{a}$ whose images generate $A_{\mathfrak{m}_i}\mathfrak{a}$, $1 \leq i \leq r$. Let $\mathfrak{a}_0 = (x_0, \ldots, x_t)$; we observe that $\mathfrak{a}_0$ and $\mathfrak{a}$ have the same extension in $A_\mathfrak{m}$ for al maximal ideals $\mathfrak{m}$ (since they do in the finite number of ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_{r+s}$ and their common extension is the whole of the ring $A_\mathfrak{m}$ in every other ideal $\mathfrak{m}$), therefore, by proposition 3.8 we deduce $\mathfrak{a} = \mathfrak{a}_0$; in particular, $\mathfrak{a}$ is finitely generated and since we chose an arbitrary ideal, A is necessarily Noetherian.

7.10

In chapter 2, exercise 6, we deduced that $M[x] \simeq M \otimes_A A[x]$. Since M and $A[x]$ are Noetherian A-modules (by assumption and Hilbert's Basis Theorem respectively), $M \oplus A[x]$ will be a Noetherian A-module. We also observe that there is a surjective map $M \oplus A[x] \twoheadrightarrow M \otimes_A A[x]$ (the natural projection $(m, a(x)) \mapsto (m \otimes_A a(x))$), hence $M \otimes_A A[x]$ will also be Noetherian by Proposition 7.1. This completes the proof.

7.11

It's not necessary that A is Noetherian. Consider, for example, $A = \prod_{i=1}^{\infty} \mathbb{Z}/2\mathbb{Z}$, a direct product of infinitely many copies of the field $\mathbb{F} = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. The strictly ascending chain of ideals

$$\mathbf{0} \subset \mathbb{F} \times \mathbf{0} \subset \mathbb{F} \times \mathbb{F} \times \mathbf{0} \ldots$$

shows that A is not Noetherian. It's also evident that every element of A is idempotent (A is Boolean).

Given any prime ideal $\mathfrak{p}$ of A, $A_{\mathfrak{p}}$ is a local integral domain with maximal ideal $\mathfrak{p}_{\mathfrak{p}}$. We claim that $\mathfrak{p}_{\mathfrak{p}} = 0$; indeed, given any non-zero element $x \in A_{\mathfrak{p}}$, we have $x(1-x) = 0$, therefore $1 - x = 0$, a conclusion that contradicts the primality of $\mathfrak{p}$. Therefore, $\mathfrak{p}_{\mathfrak{p}} = 0$ and $A_{\mathfrak{p}}$ is a field (which is Noetherian) for every prime ideal $\mathfrak{p}$ of A.

7.12

By exercise 1 (the Cohen criterion), we may examine just examine ascending chains of prime ideals in A. Let

$$\mathfrak{p}_1 \subseteq \mathfrak{p}_2 \subseteq \cdots \subseteq \mathfrak{p}_n \subseteq \ldots$$

be one. Since the induced map $f^* : \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(A)$ is surjective (by faithful flatness), the above chain gives rise to a chain

$$\mathfrak{q}_1 \subseteq \mathfrak{q}_2 \subseteq \cdots \subseteq \mathfrak{q}_n \subseteq \ldots,$$

where $\mathfrak{p}_i = f^*(\mathfrak{q}_i)$. Since the latter chain terminates (B is assumed Noetherian), the former must too. Thus A is Noetherian, as desired.

7.13

The fibre of f^* at $\mathfrak{p} \in \operatorname{Spec}(B)$ is of course $\operatorname{Spec}(k(\mathfrak{p}) \otimes_A B) = \operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$. Since f is of finite type and A is Noetherian, B will also be Noetherian, hence so will $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$ be. This means that $\operatorname{Spec}(B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$ will be a Noetherian subspace of B, as desired.

Nullstellensatz, strong form

7.14

We will essentially repeat the hint of the book; it constitutes a full proof. It is of course clear that $r(\mathfrak{a}) \subseteq I(V)$. Conversely, if $f \notin r(\mathfrak{a})$, then there is a prime ideal $\mathfrak{p}$ that contains $r(\mathfrak{a})$ but not f. Let $\overline{f}$ be the image of f under the natural projection map $A \longrightarrow A/\mathfrak{p}$ and let $C = B_f = B[1/\overline{f}]$. If $\mathfrak{m}$ is a maximal ideal of C, then $C/\mathfrak{m} \simeq k$ (by the corollary to proposition 7.9; note that C is a finitely generated k-algebra, therefore the conditions of the lemma are satisfied). The images x_i, $1 \leq i \leq n$ in C of the generators t_i, $1 \leq i \leq n$ define a point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in k^n$ which belong to the variety V, but $f(\mathbf{x}) \neq 0$. Therefore, we also have $I(V) \subseteq r(\mathfrak{a})$. This completes the proof that $r(\mathfrak{a}) = I(V)$, as desired.

7.15

Under the conditions of the problem, we have the following:

(i) $\Rightarrow$ (ii) Since A is flat over itself, so is $A^n = \oplus_{i=1}^n A$.

(ii) $\Rightarrow$ (iii) Let $i : \mathfrak{m} \longrightarrow A$ be inclusion, which is, in particular, injective. If M is flat, then the map $i \otimes_A \mathrm{id} : \mathfrak{m} \otimes_A M \longrightarrow A \otimes_A M$ will also be injective.

(iii) $\Rightarrow$ (iv) If M is flat, then the exact sequence $0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$ gives rise to the exact sequence $0 \longrightarrow \mathfrak{m} \otimes_A M \longrightarrow A \otimes M \longrightarrow k \otimes_A M \longrightarrow 0$, hence by definition, $\mathrm{Tor}_1^A(k, M) = 0$.

(iv) $\Rightarrow$ (i) For this last part, we merely reproduce the book's hint; it constitutes a full proof. Let $x_1, x_2, \dots, x_n$ be the elements of M whose images in $M/\mathfrak{m}M$ are a basis for this vector space (we consider, of course, $M/\mathfrak{m}M$ as a module over the field $A/\mathfrak{m}$). By (2.8), the x_i generate M over A. Let F be the free module A^n with canonical basis $e_1, e_2, \dots, e_n$ and define $\phi : F \longrightarrow M$ by $\phi(e_i) = x_i$; let E be the kernel of this map. Then, the exact sequence

$$0 \longrightarrow E \longrightarrow F \longrightarrow M \longrightarrow 0$$

yields, by the given condition, the exact sequence

$$0 \longrightarrow k \otimes_A E \longrightarrow k \otimes_A F \stackrel{1\otimes_A \phi}{\longrightarrow} k \otimes_A M \longrightarrow 0,$$

where $k\otimes_A F, k\otimes_A M$ are vector spaces of the same dimension over k. It follows that $1\otimes_A \phi$ is an isomorphism, hence its kernel $k \otimes_A E$ must vanish. Since E is finitely generated, as a submodule of the Noetherian space F, and A is a local space, we deduce by chapter 2, exercise 3, that $E = 0$. This implies that M is isomorphic to a free module, hence it's free, as desired.

7.16

Under the conditions of the problem, we have the following:

(i) $\Rightarrow$ (ii) If M is a flat A-module, then, since flatness is a local property, $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for all prime ideals $\mathfrak{p}$ of A. But since $A_\mathfrak{p}$ is a local ring, exercise 15 implies that $M_\mathfrak{p}$ is a free A-module.

(ii) $\Rightarrow$ (iii) O.K.

(iii) $\Rightarrow$ (i) Since $A_\mathfrak{p}$ is a local ring, the given condition is equivalent to flatness of $M_\mathfrak{p}$ for all maximal ideals $\mathfrak{m}$ of A, which is equivalent to $M_\mathfrak{m}$ being flat for all maximal ideals $\mathfrak{m}$ by exercise 15, hence to M being flat since flatness is a local property.

We conclude with the following charming epigram: "flat = locally free".

7.17

The proof that every submodule of a Noetherian module has primary decomposition follows in exactly the same fashion that propositions (7.11) and (7.12) follow; the proofs are in the book.

7.18

We have the following:

(i) $\Leftrightarrow$ (ii) This equivalence follows from proposition 7.17.

(ii) $\Leftrightarrow$ (iii) Consider the mapping $\phi_x : M \twoheadrightarrow xM$ given by $m \mapsto xm$, which yields that $M/\mathfrak{p} = M/\mathrm{Ann}(x) \simeq xM$, a submodule of M.

Conversely, if $A/\mathfrak{p}$ is isomorphic to a submodule N of M, then there is an injection $\phi : A/\mathfrak{p} \longrightarrow M$. The element $x = \phi(1)$ of M satisfies $\mathrm{Ann}(x) = \mathfrak{p}$.

For the least part, we use induction on the number n of generators of M. If $n = 0$, then $M = 0$ and the result holds vacuously, so assume that $n > 0$ and the result holds for modules with less than n generators. Let $\mathfrak{p}$ be a prime belonging to 0. There's a submodule N of M such that $A/\mathfrak{p} \simeq M/N$ (since A is Noetherian, therefore so is $a/\mathfrak{p}$), hence $0 \subset N \subset M$ and N has fewer generators than M; now the inductive hypothesis completes the proof that there is a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,$$

in which each quotient M_i/M_{i-1} is of the form $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A.

7.19

Any such decomposition of $\mathfrak{a}$ is a minimal decomposition of primary ideals (since A is Noetherian), hence the uniqueness of the set of associated ideals of $\mathfrak{a}$.

7.20

We have the following:

(i) Since $\mathcal{F}$ contains all open subsets of X and is closed under complements, it also contains all closed subsets of X. Therefore, we may equivalently describe $\mathcal{F}$ as the minimal subspace of X that contains all open and closed subsets of X and is closed under finite intersections and unions. Therefore, any element of $\mathcal{F}$ will necessarily occur as the intersection of the union of a finite number of open sets with the union of a finite number of closed sets; hence any $E \in \mathcal{F}$ can be written as $E = U \cap C$, where U, C are open and closed in X respectively. Conversely, it's obvious that any set of the above form will belong to $\mathcal{F}$.

(ii) If E contains an open subset O of X, then given any $x \in X$ and any open neighborhood N around x, we will have $E \cap N \neq \varnothing$, by the irreducibility of X. Therefore, there is an element of E in any open neighborhood of X, or, equivalently, E is dense in X.

Conversely, if $E \in \mathcal{F}$ is dense in X, and $E = U \cap C$ (we keep the previous notation, of course), then given any open subset V of X, $U \cap V \neq \varnothing$ and $V \cap U$ is open in X. Therefore may restrict our attention to open subsets V contained in U. Assume that there is no open set inside E. We can assume $V \nsubseteq C$; otherwise we obtain a contradiction. But, if $x \in (X - C) \cap V$, we have $x \in N$ such that $N \subseteq X - C$ (since C is closed) and $N' = U \cap N$ which will have empty intersection with E, a contradiction since N' is a neighborhood of x and E is dense. Therefore, E contains at least one open set of X.

7.21

Let $E \in \mathcal{F}$. Then, $E \cap X_0$ is a constructible set in the irreducible space X_0 and therefore exercise 2(ii) implies that $E \cap X_0 \neq X_0$ or otherwise $E \cap X_0$ contains a non-empty open subset of X_0.

For the converse, we use the hint. Assume that $E \notin \mathcal{F}$; this implies that the collection $\mathcal{G}$ of closed subsets $X' \subset X$ such that $E \cap X' \notin \mathcal{F}$ is non-empty (it contains X) and thus has a minimal element X_0, since X is Noetherian. Let $X = Y_1 \cup \cdots \cup Y_l$, where each Y_i is irreducible. Then,

$$E \cap X_0 = (E \cap Y_1) \cup \cdots \cup (E \cap Y_k),$$

and by the minimality of X_0 each $E \cap Y_i$ belongs to $\mathcal{F}$ and thus $E \cap X_0 \in \mathcal{F}$, lest $k = 1$ and X_0 is irreducible. Therefore, by the hypothesis, we have the following two conditions:

(i) Assume that $\overline{E \cap X_0} \neq X_0$; then

$$E \cap (\overline{E \cap X_0}) \in \mathcal{F},$$

by the minimality of X_0. But $E \cap (\overline{E \cap X_0}) = E \cap X_0$, which yields $E \cap X_0 \in \mathcal{F}$, an absurdity.

(ii) On the other hand, assume that $E \cap X_0$ contains a non-empty open subset of X_0; denote that by $U_0 = U \cap X_0$, where U is some subset open in X. Then, $U_0 \in \mathcal{F}$ and the complement of U_0 in $E \cap X_0$ is in the complement $U^c \cap X_0$, which is in $\mathcal{F}$. More precisely, $E \cap X_0 = (U \cap X_0) \cup ((E - U) \cap X_0) = (U \cap X_0) \cup (E \cap (U^c \cap X_0)) \in \mathcal{F}$, a contradiction.

We deduce that $E \in \mathcal{F}$, as desired.

7.22

Let E be open and X_0 be closed and irreducible. Then $E \cap X_0$ is open on X_0.

Conversely, assume that for all closed, irreducible subsets of X_0 we have $E \cap X_0$ or $E \cap X_0$ contains an open subset of X_0. We proceed as in exercise 21, by assuming that E is not open in X_0 and arriving at a contradiction. Let $\mathcal{G}$ be the collection of all closed subsets X' of X for which $E \cap X'$ is not open in X'. Then, $X \in \mathcal{G}$, thus $\mathcal{G}$ must have a minimal element X_0; this is, as before, irreducible. Again, we treat two cases.

(i) The first is $E \cap X_0 = \varnothing$, but since $\varnothing$ is open in X_0, this implies $X_0 \notin \mathcal{G}$, an immediate contradiction.

(ii) Thus $E \cap X_0$ must contain at least one open subset of X_0, say $U \cap X_0$, where U is open in X. Now $U^c \cap X_0$ is a closed subset of X_0, thus $E \cap (U^c \cap X_0)$ is open in $U^c \cap X_0$; write this as $V \cap (U^c \cap X_0)$ for V open in X. Finally, $E \cap X_0 = (U \cap X_0) \cup (E \cap (U^c \cap X_0)) = (U \cap X_0) \cup (V \cap (U^c \cap X_0)) = (U \cup V) \cup X_0$, and this implies $X_0 \notin \mathcal{G}$, an absurdity. Therefore, $\mathcal{G}$ is necessarily empty, or E is open in X, as desired.

7.23

We will roughly repeat the hint of the book; it constitutes a full solution. Let E be a constructible set in Y; we may assume that $E = U \cap C$, where U, C are open and closed respectively. Let $C = V(\mathfrak{b}) = \operatorname{Spec}(B/\mathfrak{b})$, so by replacing B with $B/\mathfrak{b}$ and f with the composite mapping $A \longrightarrow B \longrightarrow B/\mathfrak{b}$, we may assume that E is open in Y. But Y is Noetherian, hence quasi-compact (in the sense of the book), so Y is covered by a finite number of sets of the form $Y_g = \operatorname{Spec}(B_g)$. By replacing B with one of the rings B_g, we may assume $E = Y$. That is it is enough to show that $f^*(Y)$ is constructible in X.

We shall employ the criterion of exercise 21. Let $X_0 = V(\mathfrak{p}) = \operatorname{Spec}(A/\mathfrak{p})$ be an irreducible closed set in X such that $f^*(Y) \cap X_0$ is dense in X_0; we must show that $f^*(X_0) \cap X_0$ contains a non-empty open subset of X_0. Now $f^*(Y) \cap X_0 = f^*(f^{*-1}(X_0)$, and $f^{*-1}(X_0) = f^{*-1}(V(\mathfrak{p})) = V(\mathfrak{p}^e) = \operatorname{Spec}(B/\mathfrak{p}^e) = \operatorname{Spec}((A/\mathfrak{p}) \otimes_A B)$.

But this restricted map is induced by the map $f : A/\mathfrak{p} \longrightarrow (A/\mathfrak{p}) \otimes_A B$. By assumption, $\overline{f}^*$ has a dense image and thus by chapter 1, exercise 21, f is injective. So we may assume that A is an integral domain and f is injective.

Let $Y_1, Y_2, \ldots, Y_n$ be the irreducible components of Y; it suffices to show that $f^* Y_i$ contains an open set of X, for some i. Indeed, since X is now assumed to be irreducible, we may select Y_i such that $f^*(Y_i)$ is dense in X. Write $Y_i = V(\mathfrak{q}_i)$, and replace B with $B/\mathfrak{q}_i$. By the density of $f^*(Y_i)$, the map $A \longrightarrow B/\mathfrak{q}_i$ is still injective, and so we are reduced to the case in which A and B are both integral domains and f is injective of finite type. We wish to show that $f^*(Y)$ contains an open subset of X.

By exercise 21, there is $s \neq 0$ in A such that every map of A into an algebraically closed field Ω which does not kill s extends to B. We claim that $f^*(Y)$ contains the nonempty open set X_s. Indeed, let $\mathfrak{p}$ be a prime ideal not containing s, and let ϕ be the composite map

$$A \longrightarrow A/\mathfrak{p} \longrightarrow k(\mathfrak{p}) \longrightarrow \Omega,$$

where $k(\mathfrak{p})$ is of course the fraction field of $A/\mathfrak{p}$, and Ω is any algebraic closure of it. Obviously ϕ does not vanish at s precisely because $s \notin \mathfrak{p}$, so ϕ extends to a map $\overline{\phi} : B \longrightarrow \Omega$ and $\mathfrak{q} = \ker(\overline{\phi})$ is a prime ideal of ϕ, and $\mathfrak{q} \cap A = \mathfrak{p}$ simply because $\overline{\phi}$ restricts to ϕ. So $\mathfrak{p} \in f^*(Y)$, whence $X_s \subseteq f^*(Y)$.

7.24

We merely reproduce the hint of the book; it constitutes a full proof. If f^* is an open mapping, then f has the going-down property (by chapter 5, exercise 10). Conversely, if f has the going-down property, then we may without loss of generality assume that $f^*(Y)$ is open in X. In this case, the going-down property asserts that if $\mathfrak{p} \in E$ and $\mathfrak{p}' \subseteq \mathfrak{p}$, then $\mathfrak{p}' \in E$; in other words, if X_0 is an irreducible closed subset of X and X_0 intersects with E, then $E \cap X_0$ is dense in X_0. By exercises 20 and 22, E is open in X.

7.25

Under the assumptions of the problem, the conclusion is straightforward; chapter 5, exercise 11 implies that f has the going-down property and exercise 24 implies that f^* is an open mapping.

7.26

Let's fix some notation first; let $F(A) = \{\overline{M}\}$ be the set of isomorphism classes of finitely generated modules over A, and $C = \mathbb{Z}\overline{M}$ be the free group generated by $F(A)$, as in the problem. Now, let $e(M', M, M'')$ be the isomorphism class of $M' + M'' - M$ in C, where $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence; let D be the group generated by those $e(M', M, M'')$. Then, $K(A) = C/D$ and $\gamma_A(M)$ is the image of M in $K(A)$. With the above notation, we have the following:

(i) Any additive function $\lambda : F(A) \longrightarrow G$ obviously factors as $\lambda = \lambda_0 \circ \gamma$, where λ_0 maps the class $\overline{(M)}$ to $\gamma(M)$. The map λ_0 is well defined because λ is additive.

(ii) In exercise 18 it is deduced that if

$$0 \subset M_0 \subset M - 1 \subset \cdots \subset M_n = M,$$

where $M_i/M_{i-1} = A/\mathfrak{p}_i$, then

$$\overline{(M)} = \sum_{i=1}^{n-1} \gamma(A/\mathfrak{p}_i).$$

Passing to $K(A)$ yields

$$\overline{(M)} = \sum_{k} \gamma(A/\mathfrak{p}_i),$$

as desired.

(iii) If A is a principal ideal domain, then we may put $\mathfrak{p}_k = (x_k), x_k \in A$, therefore, the previous construction yields $\gamma(A/\mathfrak{p}_k)$ for all $\mathfrak{p}_k$ except for at most one, say $\mathfrak{p}_{k_M}$. The map $K(A) \longrightarrow \mathbb{Z}$ that is defined by $\overline{(M)} \mapsto k$ yields the desired isomorphism.

(iv) The map $f : A \longrightarrow B$ send $\overline{(M)}$ to $\overline{f(M)}$; now the module M/A may first be regarded as a finite generated module over B and thus $\overline{f(M)}$ is its image in $K(B)$. The induced map $f_! : K(A) \longrightarrow K(B)$ satisfies by construction $f_!(\gamma_B(M)) = \gamma_A(N)$ and it's obvious that ! is a covariant functor in the sense that $(f \circ g)_! = f_! \circ g_!$.

7.27

With the notation of the previous problem, we have the following:

(i) The closure of the commutative ring $K_l(A)$ follows from the flatness of its elements and all other axioms are easily verified; in particular, the identity of addition is $\gamma_l(0)$ and the identity of multiplication is $\gamma_l(A)$.

(ii) Again, closure is satisfied by the flatness of the elements of $K_l(A)$ and the action of $K_l(A)$ on $K(A)$ is given by $\gamma_l(M)\gamma(N) = \gamma(M \otimes N)$.

(iii) In this case, $K_l(A) = K(A/\mathfrak{m}) \simeq \mathbb{Z}$, where $\mathfrak{m}$ is of course the maximal ideal of A.

(iv) We just repeat the hint for this question, but it's at any rate obvious. If M is flat and finitely generated over A, then $B \otimes_A M$ is flat and finitely generated over B; this condition guarantees that the map is well-defined. Now the definition itself suffices to make the implication $(f \circ g)^! = f^! \circ g^!$ obvious.

(v) It's obvious by the definition of $f^!$ that $f^!(f_!(x)y) = xf_!(y)$ for every $x \in K_l(A), y \in K(B)$.

Chapter 8

Artin Rings

8.1

The existence of a positive integer r_i such that $\mathfrak{p}_i^{(r_i)} \subseteq \mathfrak{q}_i$ follows immediately from the definition of $\mathfrak{p}^{(n)} = S_{\mathfrak{p}}(\mathfrak{p}^n) \subseteq \mathfrak{p}^n$, and the fact that in an Artin ring there is always some power of the radical of any ideal $\mathfrak{q}_i$ (here, this radical is $\mathfrak{p}_i$) that is contained to $\mathfrak{q}_i$. This completes this exercise.

8.2

We assume that a topological space is called *discrete* if and only if all its subspaces are closed:

(i) $\Rightarrow$ (ii) If A is Artinian, then all of its prime ideals are maximal, therefore there is a finite number of prime ideals, since the number of maximal ideals is necessarily finite (by proposition 8.3). This implies that $\mathrm{Spec}(A)$ is finite. Since every $\mathfrak{p} \in \mathrm{Spec}(A)$ is maximal, we obtain that the point $\{\mathfrak{p}\} \in \mathrm{Spec}(A)$ is closed; the above imply that $\mathrm{Spec}(A)$ is discrete, as desired.

(ii) $\Rightarrow$ (iii) O.K.

(iii) $\Rightarrow$ (i) If $\mathrm{Spec}(A)$ is discrete, then $\{\mathfrak{p}\}$, where $\mathfrak{p}$ is any prime ideal, is closed, therefore $\mathfrak{p}$ is maximal in A. This implies that the Krull dimension of A is 0, and any Noetherian ring of Krull dimension 0 is Artinian, by Theorem 8.5. This completes the proof.

8.3

We have the following:

(i) If A is Artinian, then it can be written as a finite product of local Artin rings, hence we may consider it to be local without loss of generality. The residue field $\mathbb{F} = A/\mathfrak{m}$ of A will be a finite algebraic extension of k, by the Nullstellensatz (Corollary 5.24). Since A is Artinian, it will be of finite length as an A-module, and we conclude that it will be a finite k-algebra, as desired.

(ii) The converse is obvious: the ideals of A are k-modules, thus k-vector spaces, and so they satisfy the descending chain condition and A is Artinian.

8.4

We have the following:

(i) $\Rightarrow$ (ii) If the fibres of f^* are finite, then they are also discrete subspaces of $\mathrm{Spec}(B)$, by exercise 2.

(i) $\Leftrightarrow$ (iii) The fibres $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A B)$ of f^* are discrete subspaces of $\mathrm{Spec}(B)$ if and only if they are finite (by exercise 3) $k(\mathfrak{p})$-algebras ($k(\mathfrak{p})$ being the residue field of $A_{\mathfrak{p}}$).

(iii) $\Rightarrow$ (iv) Here, the fibres of f^* will be finite, again by exercise 3.

8.5

Indeed, the number of such points cannot be greater than the dimension of A. In particular, it is finite.

8.6

Since by assumption $r(\mathfrak{q}) = \mathfrak{p}$, any chain of primary ideals $\mathfrak{q} = \mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_m = \mathfrak{p}$ will consist of $\mathfrak{p}$-primary ideals $\mathfrak{q}_i$. We note that any such chain will be of finite length (by the Noetherian condition on A) and all such lengths will be bounded (the *height* of $\mathfrak{p}$ is finite), since merging two chains by taking the intersection of their respective elements and then attaching the extra terms up to $\mathfrak{p}$ yields a new chain of $\mathfrak{p}$-primary ideals with length equal to the length of the greatest chain. Therefore, if for every chain we could find another chain of greater length, we would be able to construct a chain of infinite length, a contradiction.

By the above argument, we deduce that any maximal chain (i.e. any chain that cannot be further refined by merging) will have the same length as any other maximal chain; otherwise, merging two maximal chains of different length would yield a chain strictly finer (thus longer) than both of them, a contradiction. In fact, the length of the maximal chains will be equal to the number r in the statement of the Noether Normalization Lemma, chapter 5, exercise 16.

Chapter 9

Discrete Valuation Rings and Dedekind Domains

9.1

We see that $S^{-1}A$ is a Noetherian domain if A is one and the Krull dimension of $S^{-1}A$ is less than or equal to the dimension 1 of A. Thus if $\dim(S^{-1}A) = 0$, then $S^{-1}A$ is the field of fractions of A; otherwise, it's a Dedekind domain.

The exact sequences

$$1 \longrightarrow U_A \longrightarrow K^* \longrightarrow I \longrightarrow H \longrightarrow 0$$

and

$$1 \longrightarrow U_{S^{-1}A} \longrightarrow K^* \longrightarrow I_{S^{-1}A} \longrightarrow H_{S^{-1}A} \longrightarrow 0$$

for A and $S^{-1}A$ respectively induce explicitly the surjective map $H \longrightarrow H'$ given by $\overline{(u)} \mapsto \overline{(u/1)}$; in fact, this is merely extension of ideals.

9.2

Since equality is a local property, we may assume that A is a discrete valuation ring without loss of generality. If $\dim A = 0$, then A is a field and the proof is elementary. Otherwise, assume that $\dim A = 1$. It is always true that $c(fg) \supseteq c(f)c(g)$, without any conditions on A. For the converse, assume that the maximal ideal $\mathfrak{m}$ of A is generated by x and let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ with coefficients $a_i, b_j \in A$. Now, if $c(f) = x^sA$ and $c(g) = x^tA$, then $v(x) = 1, \min\{v(a_0), v(a_1), \ldots, v(a_n)\} = s, \min\{v(b_0), v(b_1), \ldots, v(b_m)\} = t$. Then, there exist non-negative n_0, m_0 less than n, m respectively, such that $v(a_{n_0}) = s, v(b_{m_0}) = t$ and $v(a_i) > s, v(b_j) > t$ for all $i < n_0, j < m_0$. Also, the coefficient of $x^{n_0+m_0}$ in the polynomial fg is

$$c_{n_0+m_0} = \sum_{i+j=n_0+m_0} a_ib_j,$$

where we agree that $a_i = b_j = 0$ if $i > n$ or $j > m$. Now it's easy to see that $v(a_{n_0}b_{m_0}) = s+t$ and all the other terms have valuations strictly larger than $s+t$, by the construction of n_0, m_0. Therefore, $v(c_{n_0+m_0}) = s+t$, which implies that $c(fg) \supseteq c_{n_0+m_0}A = x^{s+t}A = c(f)c(g)$. This shows the inverse inclusion and completes the proof.

9.3

We claim that a Noetherian valuation ring is a principal ideal domain (PID), hence has dimension at most 1 (non-zero prime ideals in PID's are always maximal). A Noetherian valuation ring which is not a field would then satisfy the hypotheses of Proposition 9.2 and the equivalent condition (iii) of that proposition, hence it's a discrete valuation ring (DVR). For the proof, just observe that in an arbitrary valuation ring, all finitely generated ideals are principal; given a finite list of generators for an ideal, the entire ideal is generated by the generator with the least valuation. In a Noetherian ring, this is case for all ideals, yielding the desired conclusion.

Conversely, it follows immediately from the definitions that any discrete valuation ring is Noetherian and a valuation ring.

9.4

We will construct a valuation on A. Let m be a generator of $\mathfrak{m}$. Given a nonzero element $x \in A$, we may write x uniquely in the form um^t where u is a unit in A and t is a nonnegative integer. It is clear that if such a representation exists, it is in fact unique. There is actually one, and this is shown as follows: let $t \geq 0$ be such that $x \in \mathfrak{m}^t - \mathfrak{m}^{t+1}$. Then, $x = um^t$ for some u and the fact that $x \notin m^{t+1}$ shows that u is a unit. The map $A \longrightarrow \mathbb{Z}$ given by $x \mapsto t$ is the desired discrete valuation on A.

9.5

Since M is finitely generated, M is in fact a quotient A^n/K for some $n \in \mathbb{N}$. Now, since A is Noetherian, M is torsion-free if and only if $M_\mathfrak{p}$ is torsion-free, hence if and only if $M_\mathfrak{p}$ is free (because, given $\mathfrak{p} \in \operatorname{Spec}(A)$, there exists $n_\mathfrak{p} \in \mathbb{N}$ such that $M_\mathfrak{p} \simeq A^{n_\mathfrak{p}}/K_\mathfrak{p}$ by $M_\mathfrak{p}$ being finitely generated and this module is torsion-free if and only if $K_\mathfrak{p} = 0$) for all prime ideals $\mathfrak{p}$ of A. But local freedom is equivalent to flatness over Noetherian rings, which yields the desired equivalence.

9.6

Let $\mathfrak{p}$ be a prime ideal of A; then $M_\mathfrak{p}$ is a finitely generated torsion module over the principal ideal domain $A_\mathfrak{p}$. In fact, we claim that $M_\mathfrak{p} = 0$ except for a finite number of primes. For $\operatorname{Ann}(M) \neq 0$, since M is a torsion submodule and so

$$\operatorname{Ann}(M) = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_m^{n_m},$$

where $m_i > 0$. But then

$$\operatorname{Supp}(M) = V(\operatorname{Ann}(M)) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}.$$

Now, if $M_\mathfrak{p} \neq 0$, then the structure theorem for principal ideal domains yields that

$$M_\mathfrak{p} = \bigoplus_{i=1}^{t} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}^{m_i} = \bigoplus_{i=1}^{t} (A/\mathfrak{p}^{m_i})_\mathfrak{p}.$$

Let D be this module on the right. In fact, $D_\mathfrak{p} = D$, which follows from the following lemma:

Let A be a Dedekind domain, $\mathfrak{p}, \mathfrak{q}$ nonzero prime ideals of A, and $m > 0$. Then,

$$(A/\mathfrak{p}^m)_\mathfrak{p} \simeq A/\mathfrak{p}^m$$

and

$$(A/\mathfrak{p}^m)_\mathfrak{q} = 0.$$

For the proof, note that there is an obvious map from the right-hand side to the left-hand side: $\overline{x} \mapsto \overline{x/1}$. It is injective, or equivalently $sa \in \mathfrak{p}^m$ with $s \notin \mathfrak{p}$ implies $a \in \mathfrak{p}^m$, since by unique factorization of ideals $\mathfrak{p}^m$ divides the ideal (sa) and $\mathfrak{p}$ does not divide (s), hence $\mathfrak{p}^m$ divides (a), as desired. For surjectivity, let $x \in A - \mathfrak{p}$; we wish to show that x is invertible modulo $\mathfrak{p}^m$. But $\mathfrak{p}$ is maximal, so x is already invertible modulo $\mathfrak{p}$, namely there exists $y \in A - \mathfrak{p}$ such that $1 + xy \in \mathfrak{p}$. Taking this relation to the m'th power yields $(1 + xy)^m \in \mathfrak{p}^m$, and expanding the left-hand side yields the inverse of x modulo $\mathfrak{p}^m$. For the second statement, note that since $\mathfrak{p}$ and $\mathfrak{q}$, $\mathfrak{q} \not\subseteq \mathfrak{p}$, there exists $s \in \mathfrak{p} - \mathfrak{q}$, for which s^m kills $A/\mathfrak{p}^m$. This completes the proof of the lemma.

For each $\mathfrak{p}_i \in \operatorname{Supp}(M)$, let D_i be the corresponding module D, given in the decomposition of $M_\mathfrak{p}$via the structure theorem. The lemma yields that $(D_i)_{\mathfrak{p}_j}$ is equal to $\delta_{ij}D_i$, where $\delta_{ij}$is the Kronecker delta. Now let

$$D = \bigoplus_{i=1}^{t} D_i;$$

observe that the composition

$$M \longrightarrow \bigoplus_{\mathfrak{p}\neq 0} M_\mathfrak{p} \xrightarrow{\simeq} \bigoplus_{i=1}^{m} D_i$$

is an isomorphism when localized at each non-zero prime ideal of A. Hence the original map is an isomorphism, as desired.

9.7

Suppose $\mathfrak{a} = \mathfrak{p}$ is a prime ideal in A, and let $n > 0$. Then, by the lemma we showed in exercise 9.6, $A/\mathfrak{p}^n$ is isomorphic to $(A/\mathfrak{p}^n)_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p}$, namely a quotient of the principal ideal domain, which is a principal ideal ring itself. Hence the statement holds in this case. For a general $\mathfrak{a}$, write $\mathfrak{a} = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_r^{n_r}$ (where $n_i > 0$) and note that

$$A/\mathfrak{a} = \bigoplus_{\mathfrak{p}\neq 0} (A/\mathfrak{a})_\mathfrak{p} = \bigoplus_{i=1}^{r} (A/\mathfrak{p}_i^{n_i})$$

by the previous exercise. Hence $A/\mathfrak{a}$ is a product of principal ideal rings, and thus an ideal ring itself.

9.8

As equality is a local property and sums and intersections of ideals commute with localization, we may assume that A is a discrete valuation ring without loss of generality. Then the maximal ideal $\mathfrak{m}$ of A is principal, say $\mathfrak{m} = (x)$ and every other ideal is merely (x^k) for some $k \geq 0$. In particular, say $\mathfrak{a} = (x^a), \mathfrak{b} = (x^b), \mathfrak{c} = (x^c)$, with $a, b, c \in \mathbb{N}$. Then, it's easy to check (and a standard fact from Algebraic Number Theory) that $\mathfrak{a} \cap \mathfrak{b} = (x^{\max(a,b)}$ and $\mathfrak{b} + \mathfrak{c} = (x^{\min(b,c)})$. The desired equalities of ideals then merely become $\max(a, \min(b, c)) = \min(\max(a, b), \max(b, c))$ and $\min(a, \max(b, c)) = \max(\min(a, b), \min(b, c))$. But these are trivially true, as desired.

9.9

The forward implication is clear, for if x is congruent to x_i modulo $\mathfrak{a}_i$ and to x_j modulo $\mathfrak{a}_j$, then x is congruent to both modulo $\mathfrak{a}_i + \mathfrak{a}_j$, hence x_i and x_j must be congruent modulo $\mathfrak{a}_i + \mathfrak{a}_j$.

For the other direction, consider the composition of maps

$$A \xrightarrow{\phi} \bigoplus_{i=1}^{n} A/\mathfrak{a}_i \xrightarrow{\psi} \bigoplus_{i<j} A/(\mathfrak{a}_i + \mathfrak{a}_j),$$

where $\phi(x) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n)$ and $\psi(x) = (x_1 + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n)$ has (i, j)'th component equal to $x_i - x_j + \mathfrak{a}_i + \mathfrak{a}_j$. The given proposition is equivalent to the previous diagram being an exact sequence. Note that since exactness is a local notion, we may assume that A is a discrete valuation ring, by localizing at every nonzero prime ideal of A.

Under this assumption, A has a unique maximal ideal $\mathfrak{m}$, and each $\mathfrak{a}_i$ is a power $\mathfrak{m}^{n_i}$ of $\mathfrak{m}$ for some $m_i > 0$. We may reorder the ideals so that $m_i \leq m_{i+1}$, or $\mathfrak{a}_i \supseteq \mathfrak{a}_{i+1}$ for all i's. Now let $(x_1 + \mathfrak{a}_1, \dots, x_n + \mathfrak{a}_n) \in \ker\psi$; this immediately implies that $x_i \equiv x_j (\text{mod } \mathfrak{a}_i)$ whenever $i < j$. In particular, $x_n \equiv x_1 (\text{mod } \mathfrak{a}_1)$, as well as $x_n \equiv x_2 (\text{mod } \mathfrak{a}_2)$, ... and $x_n \equiv x_{n-1} (\text{mod } \mathfrak{a}_{n-1})$, so we have solved the system with $x = x_n$.

Note that the kernel of ϕ is $\bigcap_i \mathfrak{a}_i$. In a Dedekind domain, this yields the following result: If $\mathfrak{a} \subset A$ is a nonzero ideal, and

$$\mathfrak{a} = \mathfrak{p}_1^{m_1} \dots \mathfrak{p}_n^{m_n},$$

then

$$A/\mathfrak{a} = A/(\mathfrak{p}_1^{m_1} \dots \mathfrak{p}_n^{m_n}) = A/\mathfrak{p}_1^{m_1} \times \dots \times A/\mathfrak{p}_n^{m_n},$$

independently of the last problem.

Chapter 10

Completions

10.1

By the definition of A and B, we have that:

$$A = \bigoplus_{n\geq 0} \mathbb{Z}/p\mathbb{Z}$$

and

$$B = \bigoplus_{n\geq 0} \mathbb{Z}/p^n\mathbb{Z}.$$

This implies $pA = 0$, whence

$$\hat{A} = \varprojlim_{k\geq 0} A/p^k A = \varprojlim_{k\geq 0} A/(0) = A;$$

this verifies the first assertion of the problem, namely that the completion of A with respect to the p-adic topology is in fact A.

Now, for the second claim, consider the fundamental neighborhoods of B with the p-adic topology; these are precisely $p^k B$ for $k \geq 0$. The topology induced on A by this topology on B would then have fundamental neighborhoods given by $A_k = \alpha^{-1}(p^k B)$ which is explicitly given by

$$A_k = \alpha^{-1}(0 \oplus 0 \cdots \oplus 0 \bigoplus_{n>k} \mathbb{Z}/p^n\mathbb{Z}) = \bigoplus_{n>k} \mathbb{Z}/p\mathbb{Z}.$$

Therefore, the completion of A with respect to this topology is

$$\hat{A} = \varprojlim_{k\geq 0}(A/A_k) = \varprojlim_{k\geq 0}(\mathbb{Z}/p\mathbb{Z})^k = \prod_{n\geq 0} \mathbb{Z}/p\mathbb{Z},$$

since for any countable family of modules $\{M_n\}_{n\geq 0}$ it is a standard exercise that the inverse limit of the system $\{M_0 \oplus M_1 \oplus \cdots \oplus M_n\}_{n\geq 0}$, with the obvious mappings between the sums, is merely $\prod M_n$. The above implies, in particular, that the completions of A with respect to the two different topologies fail to coincide.

For the final remark, consider the following sequence:

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{p\cdot} B \longrightarrow 0;$$

this is exact, but the completed sequence

$$\hat{A} \xrightarrow{\hat{\alpha}} \hat{B} \xrightarrow{\hat{p}\cdot} \hat{B} \longrightarrow 0$$

is not, since the kernel of the $\hat{p}\cdot$ map is $\prod_{k\geq 0} \mathbb{Z}/p\mathbb{Z}$, which is the completion of A with respect to the topology induced by B, but this does not coincide with the p-adic completion of A. Thus, p-adic completion does not commute with taking kernels, and therefore it is not a right-exact functor on the category of all $\mathbb{Z}$-modules, as desired.

10.2

The given exact sequence is explicitly the following:

$$0 \longrightarrow \bigoplus_{k>n} \mathbb{Z}/p\mathbb{Z} \longrightarrow \bigoplus_{k\geq 0} \mathbb{Z}/p\mathbb{Z} \longrightarrow \bigoplus_{n\geq k\geq 0} \mathbb{Z}/\mathbb{Z} \longrightarrow 0.$$

By the previous exercise, we have

$$\varprojlim_{k\geq 0} A/A_n = \prod_{n\geq 0} \mathbb{Z}/p\mathbb{Z}$$

and

$$\varprojlim_{n\geq 0} A_n = 0.$$

Therefore, the completed sequence cannot be exact.

However, we claim that $\varprojlim{}^1 A = 0$; this follows from the following abstract nonsense argument: let $\mathcal{C}$ be the category of inverse systems of R-modules, and $\mathcal{D}$ the subcategory of constant inverse systems of R-modules, which we regard via the obvious isomorphism to the category of R-modules. The inclusion functor $\mathcal{D} \longrightarrow \mathcal{C}$ has a left-adjoint, the left-exact functor that takes an inverse system $\{C_n\}_{n\in\mathbb{N}}$ in $\mathcal{C}$ to its first term C_1. It follows that the inclusion functor $\mathcal{D} \longrightarrow \mathcal{C}$ preserves injectives, that is, a constant inverse system whose underlying module is injective is in fact an injective object in the category of inverse systems. Now, let I be an injective module with $A \hookrightarrow I$, and consider the short exact sequence $0 \longrightarrow A \longrightarrow I \longrightarrow I/A \longrightarrow 0$ as a short exact sequence of constant inverse systems. Looking at the long exact sequence arising from derived functors $\varprojlim{}^1 \cdot$, we find $\varprojlim{}^1 A = 0$, since the map $i \longrightarrow I/A$ is surjective and $\varprojlim{}^1 I = 0$. This completes the proof of the claim.

The above implies that the long exact sequence of derived functors begins:

$$0 \longrightarrow 0 \longrightarrow \bigoplus_{n\geq 0} \mathbb{Z}/p\mathbb{Z} \longrightarrow \prod_{n\geq 0} \mathbb{Z}/p\mathbb{Z} \longrightarrow \varprojlim{}^1 A_n \longrightarrow 0,$$

hence $\varprojlim{}^1 A_n = (\prod \mathbb{Z}/p\mathbb{Z})/(\bigoplus \mathbb{Z}/p\mathbb{Z})$.

10.3

By Krull's Theorem, the submodule

$$E = \bigcap_{n=1}^{\infty} \mathfrak{a}^n M$$

is annihilated by some element of the form $1 + a$, with $a \in \mathfrak{a}$. Hence $E_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ that contain $\mathfrak{m}$ since $1 + a$ is a unit in $A_\mathfrak{m}$ if $\mathfrak{a} \subseteq \mathfrak{m}$. Therefore,

$$\bigcap_{n=1}^{\infty} \mathfrak{a}^n M = E \subseteq \bigcap_{\mathfrak{m}\supseteq\mathfrak{a}} \ker(M \longrightarrow M_\mathfrak{m}).$$

Conversely, let

$$K = \bigcap_{\mathfrak{m}\supseteq\mathfrak{a}} \ker(M \longrightarrow M_\mathfrak{m}).$$

Then, $K_\mathfrak{m} = 0$ for all maximal ideals containing $\mathfrak{a}$, which implies that $K = \mathfrak{a}K$. Hence

$$K = \mathfrak{a}K = \mathfrak{a}^2 K = \cdots = \mathfrak{a}^n = \cdots = \bigcap_{n\in\mathbb{N}} \mathfrak{a}^n M.$$

Therefore, the desired relation holds.

To prove the other claim, first note that $\hat{M} = 0$ if and only if $\hat{M} = \mathfrak{a}\hat{M}$ which is true if and only if $M = \mathfrak{a}M$, which is equivalent to

$$M = \bigcap_{n \in \mathbb{N}} \mathfrak{a}^n M = \bigcap_{n \in \mathbb{N}} \ker(M \longrightarrow M_{\mathfrak{m}}),$$

namely

$$M_{\mathfrak{m}} = 0$$

for all maximal ideals $\mathfrak{m}$ containing $\mathfrak{a}$. This is equivalent to asserting that $\mathrm{Supp}(M) \cap V(\mathfrak{a})$ contains no maximal ideals, therefore to the desired statement that $\mathrm{Supp}(M) \cap V(\mathfrak{a}) = \varnothing$.

10.4

Since x is not a zero divisor in A, the sequence

$$0 \longrightarrow A \xrightarrow{x\cdot} A \longrightarrow A/xA \longrightarrow 0$$

is exact. Since completion is an exact functor, the completed sequence

$$0 \longrightarrow \hat{A} \xrightarrow{\hat{x}\cdot} \hat{A} \longrightarrow \hat{A}/\hat{x}\hat{A} \longrightarrow 0$$

is also exact and this implies that $\hat{x}$ is not a zero divisor, as desired.

However, this does not yield the second assertion, namely that "A is an integral domain $\Rightarrow$ $\hat{A}$ is an integral domain". A counterexample to the above statement, due to Nagata and appearing in Eisenbud's *Commutative Algebra*, pages 187-188, is the following: let $R = k[x, y]$ and $\mathfrak{m} = (x, y)$; the $\mathfrak{m}$-adic completion of R is of course $\hat{R} = k[[x, y]]$. Then, consider $A = k[x, y]/(y^2 - x^2 - x^3)$ and $\overline{\mathfrak{m}} = (\overline{x}, \overline{y}) \subset A$. We claim that A is an integral domain, but its $\overline{\mathfrak{m}}$-adic completion is not. For the first claim, note that $(y^2 - x^2 - x^3)$ is irreducible, hence it is prime, since R is a unique factorization domain. Therefore, A is in fact an integral domain. However, $\hat{A} = k[[x, y]]/(y^2 - x^3 - x^2)$ is not. Let $f \in k[[x, y]]$ be such that $f^2 = 1 + x$; an example of such a power series can be obtained in an elementary fashion (by equating coefficients, which yields

$$f = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots \in k[[x]] \subset k[[x, y]]).$$

For this element, $y^2 - x^2 - x^3 = (y - xf)(y + xf)$, which shows that $y^2 - x^2 - x^3$ is not prime in $k[[x, y]]$. Therefore, $\hat{A}$ is not an integral domain.

10.5

We shall avoid following the hint of the book; it's rather convoluted and an easier proof is possible. Since M is assumed finitely generated over A, corollary (10.13) implies that

$$\hat{M} \simeq \hat{A} \otimes_A M,$$

for any completions of A and M. Therefore,

$$(\hat{M}^{\mathfrak{a}})^{\mathfrak{b}} \simeq \hat{A}^{\mathfrak{b}} \otimes_A (\hat{A}^{\mathfrak{a}} \otimes_A M) \simeq (\hat{A}^{\mathfrak{b}} \otimes \hat{A}^{\mathfrak{a}}) \otimes_A M,$$

while

$$\hat{M}^{\mathfrak{a}+\mathfrak{b}} \simeq \hat{A}^{\mathfrak{a}+\mathfrak{b}} \otimes_A M.$$

It thus clearly suffices to show that

$$(\hat{A}^{\mathfrak{a}})^{\mathfrak{b}} \simeq \hat{A}^{\mathfrak{a}+\mathfrak{b}}.$$

Indeed, the obvious inclusions

$$(\mathfrak{a} + \mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a} + \mathfrak{b})^n$$

yield that

$$\varprojlim_n A/(\mathfrak{a}^n A + \mathfrak{b}^n A) \simeq \varprojlim_n A/(\mathfrak{a}+\mathfrak{b})^n A = \hat{A}^{\mathfrak{a}+\mathfrak{b}},$$

while the isomorphism

$$\varprojlim_m (\varprojlim_n A/(\mathfrak{a}^n A + \mathfrak{b}^m A)) \simeq \varprojlim_n A/(\mathfrak{a}^n + \mathfrak{b}^n)A$$

yields that

$$\hat{A}^{\mathfrak{a}} \otimes_A \hat{A}^{\mathfrak{b}} \simeq \hat{A}^{\mathfrak{a}+\mathfrak{b}},$$

as desired.

10.6

It clearly suffices to show that given any maximal ideal $\mathfrak{m}$, $\mathfrak{a} \subset \mathfrak{m}$ if and only if $\mathfrak{m}$ is closed in the $\mathfrak{a}$-adic topology. Indeed, if $\mathfrak{a} \subset \mathfrak{m}$, then

$$A - \mathfrak{m} = \bigcup_{x \in A - \mathfrak{m}} x + \mathfrak{a},$$

hence every point in $A - \mathfrak{m}$ has a neighborhood around it, implying that $A - \mathfrak{m}$ is open in the $\mathfrak{a}$-adic topology.

Conversely, let $\mathfrak{m}$ be closed in the $\mathfrak{a}$-adic topology. Then there exists a positive integer n such that $1 + \mathfrak{a}^n \subset A - \mathfrak{m}$. Since $\mathfrak{m}$ is maximal, we must have either $\mathfrak{a}^n \subset \mathfrak{m}$ or $\mathfrak{a}^n + \mathfrak{m} = (1)$. But the latter contradicts the choice of n, hence $\mathfrak{a}^n \subset \mathfrak{m}$, and since $\mathfrak{m}$ is prime, $\mathfrak{a} \subset \mathfrak{m}$, as desired.

10.7

Let $\hat{A}$ be faithfully flat over A. Then, for any finitely generated module M, the natural map $M \longrightarrow \hat{M}$ is injective; in particular, this is true for $M = A/\mathfrak{m}$, where $\mathfrak{m}$ is any maximal ideal of A. Thus,

$$\ker(A/\mathfrak{m} \longrightarrow \hat{A}/\hat{\mathfrak{m}}) = \bigcap_{n \geq 0} \mathfrak{a}^n (A/\mathfrak{m}) = 0.$$

If $\mathfrak{a} \not\subseteq \mathfrak{m}$, then there exists $a \in \mathfrak{a} - \mathfrak{m}$, which is a unit modulo $\mathfrak{m}$. Hence $\mathfrak{a}^n(A/\mathfrak{m}) = A/\mathfrak{m}$ for all n, which implies that the kernel is all of $A/\mathfrak{m}$, a contradiction. Hence we must have $\mathfrak{a} \subset \mathfrak{m}$. Since $\mathfrak{m}$ was arbitrary, $\mathfrak{m} \subset \mathfrak{J}_A$, which means that A is Zariski.

Conversely, let A be Zariski, with its topology being given by some $\mathfrak{a} \subset \mathfrak{J}_A$ and let $\mathfrak{m}$ be a maximal ideal. Since $\mathfrak{a} \subseteq \mathfrak{m}$, we have that the above equality is still true, and thus $A/\mathfrak{m}$ injects into $\hat{A}/\hat{\mathfrak{m}}$. In particular, $\hat{\mathfrak{m}} \neq (1)$, so $\hat{A}$ is faithfully flat over A, by chapter 3, exercise 16(iii).

10.8

We see that B is local, its unique maximal ideal being A.

Moreover, B is Zariski (since the maximal ideal topology is induced by an ideal contained in the Jacobson radical, since B is local) and thus the maximal ideal completion of B, which is C, is faithfully flat over B by the previous exercise.

10.9

Assume inductively that we have constructed $g_k(x), h_k(x) \in A[x]$ such that $f(x) - g_k(x)h_k(x) \in \mathfrak{m}^k[x]$ (the case $k = 1$ being the hypothesis of the problem). Since $\overline{g}$ and $\overline{h}$ are coprime, there exist for each $p \leq n$ polynomials a_p of degree no more than $n - r$ and b_p of degree no more than r such that

$$\overline{a}_p \overline{g}_k + \overline{b}_p \overline{h}_k = \overline{x}^p$$

or,

$$a_p g_k + b_p h_k = r_p(x) \in \mathfrak{m}[x].$$

Letting $q(x) = \sum m_p x^p$, where $m_p \in \mathfrak{m}$, we have

$$f(x) - g_k(x)h_k(x) = \sum m_p(a_p(x)g_k(x) + b_p(x)h_k(x) - r_p(x)).$$

Let $g_{k+1}(x) = g_k(x) + \sum m_p b_p(x), h_{k+1}(x) = h_k(x) + \sum m_p a_p(x)$; then, $g_{k+1} \equiv g_k(\bmod \mathfrak{m}^{k+1})$ and likewise for h_{k+1}. Also, manipulation yields

$$f(x) - g_{k+1}(x)h_{k+1}(x) = \sum m_p r_p(x) - \sum m_p b_p(x) \sum m_s a_s(x) \in \mathfrak{m}^{k+1}[x].$$

Thus we have for each k polynomials $g_k(x)$ and $h_k(x)$ of degrees r and $n-r$ respectively such that $f - g_k h_k \equiv 0(\bmod \mathfrak{m}^{k+1})$. The coefficients of these polynomials form Cauchy sequences in A, hence converge to an element of A, so we have limit polynomials $g(x), h(x) \in A[x]$. These are the polynomials we are looking for: g and h are lifts of $\overline{g}$ and $\overline{h}$ and

$$f(x) - g(x)h(x) \in \bigcap_{n \geq 0} \mathfrak{m}^n[x] = 0,$$

as desired.

10.10

We have the following:

(i) This is immediate, by letting $\overline{g}(x) = \overline{x} - \alpha$. Note that the root must be simple, so as to ensure that $\overline{g}$ and $\overline{h}$ are coprime. The fact that $a \equiv \alpha(\bmod \mathfrak{m})$ is obvious, since a and α coincide in $A/\mathfrak{m}$.

(ii) This follows easily from (a), by considering the local ring $\mathbb{Z}_7$ and observing that the monic polynomial $\overline{f}(x) = \overline{x}^2 - 2$ has a simple root in $\mathbb{Z}/7\mathbb{Z}$, which lifts back to $\mathbb{Z}_7$.

(iii) This again follows from (i) in the same fashion (ii) did (note of course that the ring os power series in two variables is just the completion of the polynomial ring in one variable only).

10.11

A counterexample is furnished by letting A to be the ring of germs of all C^∞ functions in variable x at $x = 0$; this is not a Noetherian ring. However, every power series can be regarded as the Taylor expansion of some C^∞ map, therefore the completion $\hat{A}$ is not Noetherian.

10.12

The natural inclusion $A \longrightarrow A[[x_1, x_2, \ldots, x_n]]$ factors through $A[x_1, x_2, \ldots, x_n]$ as follows

$$A \longrightarrow A[x_1, x_2, \ldots, x_n] \longrightarrow A[[x_1, x_2, \ldots, x_n]],$$

since $A[[x_1, x_2, \ldots, x_n]]$ is the $(x_1, x_2, \ldots, x_n)$-adic completion of the polynomial ring $A[x_1, x_2, \ldots, x_n]$. The first mapping above is flat (by the solution to chapter 2, exercise 5) and the second one is faithfully flat, by chapter 1, exercise 5(v.) and the definition of faithful flatness. Therefore, $A[[x_1, x_2, \ldots, x_n]]$ is a faithful A-algebra, as desired.

Chapter 11

Dimension Theory

11.1

By (11.18) we have $\dim A_{\mathfrak{m}} = n-1$. Now

$$\mathfrak{m}/\mathfrak{m}^2 \simeq (x_1, x_2, \ldots, x_n)/(x_1, x_2, \ldots, x_n)^2 + (f)$$

and it has dimension $n-1$ if and only if $f \notin (x_1, x_2, \ldots, x_n)^2$, and hence if and only if the variety $f(x) = 0$ is non-singular. Therefore, P is non-singular, if and only if $A_{\mathfrak{m}}$ is a regular local ring, as desired.

11.2

Since $k[[t_1, t_2, \ldots, t_n]]$ is the $(t_1, t_2, \ldots, t_d)$-adic completion of $k[t_1, t_2, \ldots, t_n]$ the map $t_i \mapsto x_i$ is injective. The fact that A is also a finitely-generated module over $k[[t_1, t_2, \ldots, t_d]]$ follows from proposition 10.24.

11.3

The result follows easily from the following: if $\overline{k}$ is an algebraic closure of k, then 11.25 holds for $\overline{k}$, and $\overline{k}[x_1, x_2, \ldots, x_n]$ is integral over $k[x_1, x_2, \ldots, x_n]$. Now Lemma 11.26 guarantees that the result will also hold for $k[x_1, x_2, \ldots, x_n]$, as desired.

11.4

We merely repeat the discussion of the book; there is little else to be proved. Let k be a field and $A = k[x_1, x_2, \ldots, x_n, \ldots]$ be a polynomial ring over k in countably many indeterminates. Let $\{m_n\}_{n \in \mathbb{N}}$ be an increasing sequence positive integers such that $m_{i+1} - m_i > m_i - m_{i-1}$ for all $i > 1$. Let $\mathfrak{p}_i = (x_{m_i+1}, x_{m_i+1}, \ldots, x_{m_i+1})$ and let S be the complement in A of the union of the ideals $\mathfrak{p}_i$. Then S is an multiplicatively closed set and the maximal ideals of $S^{-1}A$ are easily seen to be precisely $S^{-1}\mathfrak{p}_i$; in particular, the conditions of chapter 7, exercise 9 are fulfilled and $S^{-1}A$ is Noetherian. We conclude that, since the height of each $S^{-1}\mathfrak{p}_i$ is $m_{i+1} - m_i$ (and this sequence is strictly increasing), $\dim A = \infty$, as desired.

11.5

Let γ denote the usual map from the set of all finitely generated modules over A to its Grothendieck group $K(A)$, sending any element M to its class (M). Given the universal property of $K(A)$, Theorem 11.1 can be reformulated as follows: if λ_0 is any homomorphism from $K(A)$ to $\mathbb{Z}$ and we let the Poincaré series be defined by:

$$P(M, t) = \sum_{n=0}^{\infty} \lambda_0((M_n))t^n,$$

then

$$P(M,t) = \frac{f(t)}{\prod_{i=1}^{s}(1-t^{k_i})},$$

where $f(t) \in \mathbb{Z}[t]$.

11.6

We follow the hint of the book; it largely constitutes a proof. Let $f : A \longrightarrow A[x]$ be the natural embedding; consider the fibre of $f^* : \mathrm{Spec}(A[x]) \longrightarrow \mathrm{Spec}(A)$ over a prime ideal $\mathfrak{p}$ of A; this is naturally identified with $\mathrm{Spec}(k(\mathfrak{p}) \otimes_A A[x])$, where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}_\mathfrak{p}$ is the residue field at $\mathfrak{p}$ and $\dim k[x] = 1$, since $k[x]$ is a PID. The above observations and chapter 4, exercise 7(ii) clearly imply the desired result, namely that

$$\dim A + 1 \leq \dim A[x] \leq 2\dim A + 1.$$

11.7

Again, the hint provided by the book almost amounts to a solution and, of course, we follow it. Let $\mathfrak{p}$ be a prime ideal of height m in A. Then, there exist $a_1, a_2, \ldots, a_m \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime ideal belonging to $\mathfrak{a} = (a_1, a_2, \ldots, a_n)$. By chapter 4, exercise 7, $\mathfrak{p}[x]$ is a minimal prime ideal of $\mathfrak{a}[x]$ and therefore the height of $\mathfrak{p}[x] \leq m$. However, a chain of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_m = \mathfrak{p}$ induces a chain $\mathfrak{p}_0[x] \subset \mathfrak{p}_1[x] \subset \cdots \subset \mathfrak{p}_m[x] = \mathfrak{p}[x]$ (since A is Noetherian), and therefore the height of $\mathfrak{p}[x]$ is at least m. We deduce that the heights of $\mathfrak{p}$ and $\mathfrak{p}[x]$ are equal. By the solution of exercise 6, $\dim A[x] \geq \dim A + 1$ and the symmetric argument (note that now we have the inverse direction too) implies $\dim A[x] \leq \dim A + 1$. The desired result, $\dim A[x] = \dim A + 1$, follows.

By induction, the above claim generalizes to $\dim A[x_1, x_2, \ldots, x_n] = n + \dim A$.